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Probability Theory

Relatively compact criteria for Hilbert valued random fields on abstract Wiener space

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Abstract

In terms of the compact embedding theorems in finite dimensional Sobolev spaces, conditions are given under which Hilbert valued random fields on abstract Wiener space are relatively compact in some L^p -space. To cite this article: X. Zhang, C. R. Acad. Sci. Paris, Ser. I 342 (2006).

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Résumé

Critères de compacité relative pour des champs définis sur des espaces de Wiener abstraits et à valeurs dans un espace de Hilbert. Nous obtenons un nouveau critère pour qu'une famille de l'espace $L^p(X, B)$, définie sur un espace de Wiener et à valeurs dans un espace de Banach *B*, soit compacte. La démonstration utilise l'approximation de dimension finie et l'hypercontractivité du semi-groupe d'Ornstein–Uhlenbeck. Notre résultat est différent d'un résultat récent de Bally–Saussereau dans le sens où nous travaillons dans L^p pour tout p > 1 tandis que le résultat de Bally–Saussereau est limité à p = 2. *Pour citer cet article : X. Zhang, C. R. Acad. Sci. Paris, Ser. I 342 (2006).*

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1. Introduction and statements of main results

It is well known that in the theory of finite dimensional Sobolev spaces, classical compact embedding among different spaces (cf. [1]) are very useful tools for constructing solutions to some partial differential equations. The finiteness of the dimension is essential. Thus, in infinite dimensional Wiener–Sobolev spaces, it seems that one can not establish the analogous embedding results. However, it is still possible to find some criteria for compact families of Wiener functionals in L^p -spaces. In this direction, a first result of relative compactness criterion on Wiener space was given by Da Prato–Malliavin–Nualart in [3]. Therein, the functionals are considered from Wiener space to finite dimensional spaces, i.e., real valued random variables. Following that, by applying the finite dimensional approximation and Rellich–Kondrachov compactness theorem, we proved in [6] some other criterions for the relative compactness of real valued random variable families. Recently, Bally–Saussereau [2] applied the Wiener chaos decomposition to

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prove a relative compactness criterion in Wiener–Sobolev space, and then employed their criterion to construct the solutions for some semi-linear stochastic partial differential equations with distribution as final condition.

The aim of this Note is to give some criteria for relative compactness of Hilbert valued random fields in some L^p spaces (cf. [6,7]), which are mostly motivated by the works of Bally–Saussereau in [2]. The criteria given in the present Note are different from those of Bally–Saussereau, which strongly depend upon the Hilbert structure of L^2 . The main features are that we may discuss the relative compactness in L^p -spaces, and allow the functionals to be in fractional Wiener–Sobolev spaces.

Let (X, H, μ) be an abstract Wiener space. Namely, H is a real and separable Hilbert space, and it is continuously and densely embedded in Banach space X. Therefore, by transposition, the dual space X^* of X can be injected in Hand we have the triplet $X^* \hookrightarrow H \hookrightarrow X$. The measure μ is the Gaussian measure on $\mathcal{B}(X)$.

Let $(G, \langle \cdot, \cdot \rangle_G)$ be a separable Hilbert space. The norm is denoted by $\|\cdot\|_G$. We denote by $\mathcal{P}(G)$ the set of smooth cylindrical functions. The Ornstein–Uhlenbeck semigroup is defined by Mehler's formula for every $f \in \mathcal{P}(G)$

$$(T_t f)(x) := \int_X f(x e^{-t} + y\sqrt{1 - e^{-2t}}) \mu(dy)$$

For any p > 1, T_t can be extended a strongly continuous C_0 -semigroup of contraction on $L^p(X; G)$. In fact, it has a stronger contractivity property named Nelson's hypercontractivity (cf. [4]):

$$\|T_t f\|_{L^{p_t}(X;G)} \le \|f\|_{L^p(X;G)}, \quad p_t := 1 + e^{2t}(p-1) \ge p > 1.$$
(1)

The generator *L* of semigroup T_i is a positive self-adjoint operator on $L^2(X; G)$. For any p > 1 and $\alpha > 0$, the Sobolev space $D^p_{\alpha}(G)$ is defined by $(I - L)^{-\alpha/2}(L^p(X; G))$ and equipped with the norm $||f||_{D^p_{\alpha}(G)} := ||(I - L)^{\alpha/2}f||_{L^p(X;G)}$. For $f \in \mathcal{P}(G)$ with the form $f(x) = \sum_i F_i(\langle x, h_{i1} \rangle, \dots, \langle x, h_{ik_i} \rangle)g_i$, $F_i \in C_0^{\infty}(\mathbb{R}^{k_i})$, $h_{ij} \in H$, $g_i \in G$, the Malliavin derivative operator is defined by $Df(x) := \sum_{i,j} \partial_j F_i(\langle x, h_{i1} \rangle, \dots, \langle x, h_{ik_i} \rangle)h_{ij} \otimes g_i \in H \otimes G$. The higher derivatives can be defined similarly. For any $k \in \mathbb{N}$, Meyer's inequality states that there are two positive constants c_k , C_k such that for any $f \in \mathcal{P}(G)$ (cf. [4])

$$c_k \sum_{m=0}^k \|D^m f\|_{L^p(X; H\otimes^m G)} \leq \|f\|_{D_k^p(G)} \leq C_k \sum_{m=0}^k \|D^m f\|_{L^p(X; H\otimes^m G)}.$$

Let $(E, \|\cdot\|_E)$ be a Banach space and U an open subset(domain) of \mathbb{R}^d . For $0 < \beta < 1$ and p > 1, let $L^p(U; E)$ be the usual Banach space with respect to the Lebesgue measure du, $F^p_\beta(U; E)$ the fractional Sobolev space defined by:

$$\|f\|_{F^{p}_{\beta}(U;E)} := \|f\|_{L^{p}(U;E)} + \left(\iint_{U} \frac{\|f(u) - f(v)\|_{E}^{p}}{|u - v|^{d + \beta p}} \,\mathrm{d}u \,\mathrm{d}v\right)^{1/p} < +\infty.$$

Let *Z* be a finite dimensional subspace of X^* , \mathcal{B}_Z the Borel σ -field on *Z*, and $\tilde{\mathcal{B}}_Z = \pi_Z^{-1}(\mathcal{B}_Z)$ its inverse image on *X*, where π_Z is the projection from *X* to *Z*. The corresponding conditional expectation $\mathbb{E}[\cdot|\tilde{\mathcal{B}}_Z]$ is for short written as \mathbb{E}^Z . Let $\{Z_n, n \in \mathbb{N}\}$ be an increasing sequence of finite dimensional subspaces of X^* satisfying $\bigcup_n Z_n = H$. In the sequel, we assume that *G* is continuously and compactly embedded in some Banach space *B* with embedding constant C_B . We now state our main results as follows:

Theorem 1.1. Let U be a bounded domain of \mathbb{R}^d with strongly local Lipschitz property. Let K be a bounded subset of $L^p(U \times X, du \times \mu; G)$. Assume that for some $0 < \beta < 1, \alpha > 0$ and $p \ge q > 1$ with $\beta q > d$:

- (i) $\sup_{f \in K} \|f\|_{F^q_{\beta}(U; D^p_{\alpha}(G))} < +\infty;$
- (ii) $\|\mathbb{E}^{Z_n} f f\|_{L^p(X;C(\overline{U};B))} \to 0$ uniformly in $f \in K$ as $n \to \infty$.

Then K is relatively compact in $L^p(X; C(\overline{U}; B))$.

Theorem 1.2. Let U be an open subset of \mathbb{R}^d . Given a subset K of $L^q(U; D^p_\alpha(G))$ for some p, q > 1 and $\alpha > 0$, and assume that for some increasing compact subsets $\{U_k, k \in \mathbb{N}\}$ of U with $\bigcup_k U_k = U$

- (i) $\sup_{f \in K} \|f\|_{L^q(U, du; D^p_{\alpha}(G))} < +\infty;$
- (ii) for any $k \in \mathbb{N}$ and $h \in B_0^d(\operatorname{dis}(U_k, \partial U)), \int_{U_k} \|f(u+h, \cdot) f(u, \cdot)\|_{D_\alpha^p(G)}^q du \to 0$ uniformly in $f \in K$ as $|h| \to 0$;
- (iii) $\int_{U-U_k} \|f(u)\|_{D^p_{\omega}(G)}^q du \to 0$ uniformly in $f \in K$ as $k \to \infty$;
- (iv) $\|\mathbb{E}^{Z_n} f f\|_{L^q(U, du; L^p(X; B))} \to 0$ uniformly in $f \in K$ as $n \to \infty$.

Then K is relatively compact in $L^{q}(U, du; L^{p}(X; B))$.

In the following we only give the proof of Theorem 1.1; Theorem 1.2 can be proved similarly based on an extended Fréchet–Kolmogorov's theorem (cf. [1,7]). The condition (ii) of Theorem 1.1 can be verified for a class of SDEs or SPDEs by suitable approximation theorems. More details and several applications are provided in [7].

2. Proof of Theorem 1.1

The proof is divided into four steps.

Step 1: We may first establish the following finite dimensional embedding result (cf. [7]). Let U (resp. V) be a bounded domain of \mathbb{R}^d (resp. \mathbb{R}^n) with strongly local Lipschitz property. For p > 1 and $k \in \mathbb{N}$, let $W_k^p(V; G)$ be the usual G-valued Sobolev space. Given a bounded subset K of $F_\beta^q(U; W_k^p(V; G))$, suppose $q\beta > d$ and kp > n, then

$$\sup_{f \in K} \|f(u, z) - f(u', z')\|_G \leq C (|u - u'|^{\beta - d/q} + |z - z'|^{k - n/p})$$

for all $u, u' \in U$ and $z, z' \in V$. In particular, since G is compactly embedded in Banach space B, Ascoli–Arzela's criterion gives that K is a relatively compact subset of $C(\overline{U} \times \overline{V}; B)$.

Step 2: Since the Ornstein–Uhlenbeck semigroup T_t plays the role of 'mollifiers' in Malliavin calculus, we have $T_t f(u) \in \bigcap_{\alpha > 0} D^p_{\alpha}(G)$ for every $u \in U$. Moreover, by [5, Theorem 6.13(d)], we have

$$\|f - T_t f\|_{L^p(X;C(\overline{U};G))} \leq C t^{\alpha/2} \|f\|_{F^q_\beta(U;D^p_\alpha(G))}$$

Here and after, the constant C is independent of ε and f.

Thus, for any fixed $\varepsilon > 0$, by (i) we may choose t > 0 sufficiently small such that

$$\sup_{f \in K} \|f - T_t f\|_{L^p(X; C(\overline{U}; G))} \leqslant C\varepsilon.$$
⁽²⁾

By (ii), we have for *n* sufficiently large

$$\sup_{f \in K} \left\| \mathbb{E}^{Z_n} f - f \right\|_{L^p(X; C(\overline{U}; B))} \leqslant \varepsilon.$$
(3)

Step 3: Denote by $L_{loc}^{p,\alpha}(Z_n, dz_n; G)$ the *G*-valued functions on Z_n which are locally in the Sobolev space of exponent p and order α . Define for $\alpha \in \mathbb{N}$

$$W^p_{\alpha}(Z_n, \mu_{Z_n}; G) := \left\{ f \in L^{p,\alpha}_{\text{loc}}(Z_n, dz_n; G): \left(\partial^l f\right) \in L^p(Z_n, \mu_n; G), |l| \leq \alpha \right\}$$

where μ_n denotes the canonical Gaussian measure on Z_n . A key lemma due to Malliavin [4, p. 50 Lemma 5.2] gives that for every $u \in U$ and $n \in \mathbb{N}$

$$\mathbb{E}^{Z_n}(T_t f(u)) \in \bigcap_{\alpha>0} W^p_\alpha(Z_n, \mu_n; G),$$

and for $k \in \mathbb{N}$

$$\left\|\mathbb{E}^{Z_n}(T_t(f(u) - f(v)))\right\|_{W_k^p(Z_n,\mu_n;G)} \le C_{t,k} \left\|f(u) - f(v)\right\|_{L^p(X;G)}.$$
(4)

Let $B_a^d(r)$ denote the open ball in \mathbb{R}^d with center $a \in \mathbb{R}^d$ and radius r > 0. Set $p_t := e^{2t}(p-1) + 1$. Choose r_{ε} sufficiently large such that $\mu_n((B_0^n(r_{\varepsilon}))^c) < \varepsilon^{pp_t/(p_t-p)}$. Then

$$\mathbb{E}^{Z_n}(T_t f(u))|_{B_0^n(r_{\varepsilon})} \in \bigcap_{\alpha>0} W_{\alpha}^p(B_0^n(r_{\varepsilon}), \mu_n; G) = \bigcap_{\alpha>1} W_{\alpha}^p(B_0^n(r_{\varepsilon}), \mathrm{d}z_n; G).$$

Moreover, by (4) we also have for every k > n

$$\sup_{f\in K} \left\| \mathbb{E}^{Z_n} T_t f \right\|_{F^q_{\beta}(U; W^p_k(B^n_0(r_{\varepsilon}), \mathrm{d} z_n; G))} \leq C_{t,n,k,r_{\varepsilon}} < +\infty.$$

Therefore, by Step 1 we obtain $\{\mathbb{E}^{Z_n}(T_t f)\}_{f \in K}$ is relatively compact in $C(\overline{U} \times \overline{B_0^n(r_{\varepsilon})}; B)$. Thus, there exist finite $\{f_i\}_{i=1,...,N} \subset K$ such that for any $f \in K$, existing an f_i satisfies

$$\sup_{u\in\overline{U}}\sup_{z_n\in\overline{B_0^n(r_\varepsilon)}}\left\|\mathbb{E}^{Z_n}\left(T_t\left(f(u)-f_i(u)\right)\right)(z_n)\right\|_B\leqslant\varepsilon.$$
(5)

By Nelson's hypercontractivity (1) and Minkowski's inequality, we have

$$\|T_{t}f\|_{L^{p_{t}}(X;C(\overline{U};G))}^{p} = \left\|\sup_{u\in\overline{U}}\|T_{t}f(u,\cdot)\|_{G}\right\|_{L^{p_{t}}(X)}^{p} \leqslant C\|\|T_{t}f(u,\cdot)\|_{F_{\beta}^{q}(U;G)}\|_{L^{p_{t}}(X)}^{p}$$

$$\left(\operatorname{using}\frac{p_{t}}{q} > 1\right) \leqslant C\|T_{t}f\|_{F_{\beta}^{q}(U;L^{p_{t}}(X;G))}^{p} \leqslant C\|f\|_{F_{\beta}^{q}(U;L^{p}(X;G))}^{p} < C.$$
(6)

Then, by Hölder's inequality, (5) and (6) we get

$$\begin{aligned} \left\| \mathbb{E}^{Z_{n}} T_{t}(f-f_{i}) \right\|_{L^{p}(X;C(\overline{U};B))}^{p} \\ &\leqslant C_{B} \left\| 1_{(B_{0}^{n}(r_{\varepsilon}))^{c}} \mathbb{E}^{Z_{n}} T_{t}(f-f_{i}) \right\|_{L^{p}(X;C(\overline{U};G))}^{p} + \left\| 1_{B_{0}^{n}(r_{\varepsilon})} \mathbb{E}^{Z_{n}} T_{t}(f-f_{i}) \right\|_{L^{p}(X;C(\overline{U};B))}^{p} \\ &\leqslant C_{B} \left[\mu_{n} \left(B_{0}^{n}(r_{\varepsilon})^{c} \right) \right]^{(p_{t}-p)/p_{t}} \left\| \mathbb{E}^{Z_{n}} T_{t}(f-f_{i}) \right\|_{L^{p_{t}}(X;C(\overline{U};G))}^{p} + \varepsilon^{p} \\ &\leqslant \varepsilon^{p} + C_{B} \varepsilon^{p} \left\| T_{t}(f-f_{i}) \right\|_{L^{p_{t}}(X;C(\overline{U};G))}^{p} \leqslant C \varepsilon^{p}, \end{aligned}$$

$$(7)$$

where C_B is the embedding constant.

Step 4: By (2) (3) and (7), we achieve

$$\begin{split} \|f - f_i\|_{L^p(X;C(\overline{U};B))} &\leq \|f - \mathbb{E}^{Z_n} f\|_{L^p(X;C(\overline{U};B))} + \|\mathbb{E}^{Z_n} f - \mathbb{E}^{Z_n} f_i\|_{L^p(X;C(\overline{U};B))} + \|f_i - \mathbb{E}^{Z_n} f_i\|_{L^p(X;C(\overline{U};B))} \\ &\leq 2\varepsilon + \|\mathbb{E}^{Z_n} f - \mathbb{E}^{Z_n} T_t f\|_{L^p(X;C(\overline{U};B))} + \|\mathbb{E}^{Z_n} T_t f - \mathbb{E}^{Z_n} T_t f_i\|_{L^p(X;C(\overline{U};B))} \\ &+ \|\mathbb{E}^{Z_n} T_t f_i - \mathbb{E}^{Z_n} f_i\|_{L^p(X;C(\overline{U};B))} \\ &\leq C\varepsilon + C_B \|T_t f - f\|_{L^p(X;C(\overline{U};G))} + C_B \|T_t f_i - f_i\|_{L^p(X;C(\overline{U};G))} \leq C\varepsilon. \end{split}$$

The arbitrariness of ε produces the relative compactness of K in $L^p(X; C(\overline{U}; B))$.

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References

- [1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1978.
- [2] V. Bally, B. Saussereau, A relative compactness criterion in Wiener–Sobolev spaces and application to semi-linear stochastic PDEs, J. Funct. Anal. 210 (2) (2004) 465–515.
- [3] G. Da Prato, P. Malliavin, D. Nualart, Compact families of Wiener functionals, C. R. Acad. Sci. Paris, Sér. I 315 (1992) 1287–1291.
- [4] P. Malliavin, Stochastic Analysis, Grundlehren Math. Wiss., Springer-Verlag, Berlin, 1997.
- [5] A. Pazy, Semi-Groups of Linear Operators and Applications, Springer-Verlag, Berlin, 1985.
- [6] X. Zhang, Relatively compact sets on abstract Wiener space, Acta Math. Sinica 21 (4) (2005) 819-822.
- [7] X. Zhang, Relatively compact families of functionals on abstract Wiener space and applications, J. Func. Anal., in press.

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