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Mathematical Analysis

$\mathcal{D}$-modules on the complex projective space $\mathbb{CP}^{n-1}$ associated to a quadric ☆

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Abstract

We give a combinatorial description of regular holonomic systems on the complex projective space $\mathbb{CP}^{n-1}$ with characteristic variety the union of the zero section and the conormal bundle of a smooth quadric (equivalently: those that admit an infinitesimal action of $PO(n)$). ***To cite this article: P. Nang, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

$\mathcal{D}$-modules sur le projectif $\mathbb{CP}^{n-1}$ associé à une quadrique. Nous donnons une description combinatoire des systèmes holonômes réguliers sur l'espace projectif complexe $\mathbb{CP}^{n-1}$ dont la variété caractérisque est réunion de la section nulle et d'une quadrique lisse (de façon équivalente : ceux qui admettent une action infinitésimale de $PO(n)$). ***Pour citer cet article : P. Nang, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Une des questions fondamentales en analyse algébrique concerne la description complète des systèmes holonômes à singularités régulières. Plusieurs auteurs s'y sont interéssés notament Boutet de Monvel [1] a donné une présentation courte et élégante des $\mathcal{D}$-modules holonômes réguliers à une variable. P. Deligne [3] a donné une description combinatoire des faisceaux pervers sur $\mathbb{C}$ relativement à la stratification $\{0\}$, $\mathbb{C}\backslash\{0\}$. Ces résultats sont le point de départ de nombreux travaux sur les faisceaux pervers en dimension supérieure (voir, par exemple, [9]). Notons $X = \mathbb{C}^n$, $\widetilde{X}$ l'espace projectif $\mathbb{CP}^{n-1}$ associé, $\mathcal{D}_{\widetilde{X}}$ (resp. $\mathcal{D}_X$) le faisceau des opérateurs différentiels sur le projectif $\widetilde{X}$ (resp. X). Soit $\widetilde{X} = (\widetilde{X}\backslash\widetilde{Q}) \cup \widetilde{Q}$ une stratification de $\mathbb{CP}^{n-1}$ où $\widetilde{Q}$ est l'hypersurface définit par une forme quadratique sur $\mathbb{CP}^{n-1}$. On se propose de classifier les $\mathcal{D}_{\mathbb{CP}^{n-1}}$-modules holonômes réguliers $\mathcal{M}$ dont la variété caractéristique $\mathrm{car}(\mathcal{M})$ est contenu dans la réunion $\tilde{\Lambda}$ des fibrés conormaux aux strates de $\mathbb{CP}^{n-1}$:

$$\mathrm{car}(\mathcal{M}) \subset \tilde{\Lambda} := T^*_{(\widetilde{X}\backslash\widetilde{Q})}\tilde{X} \cup T^*_{\widetilde{Q}}\widetilde{X}. \tag{1}$$

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Ils forment une catégorie abélienne que nous noterons $\mathrm{Mod}^{\mathrm{rh}}_{\tilde{\Lambda}}(\mathcal{D}_{\mathbb{CP}^{n-1}})$. Soient $G := \mathrm{SO}(q)$ le groupe des rotations (avec q une forme quadratique non degenerée sur X) et $\overline{\mathcal{B}} := \Gamma(X, \mathcal{D}_X)^G$ l'algèbre de Weyl des opérateurs différentiels G-invariants. On note $\mathcal{B}$ le quotient de $\overline{\mathcal{B}}$ par l'ideal $\mathcal{J} \subset \overline{\mathcal{B}}$ des opérateurs nuls sur les fonctions G -invariantes. Soit θ le champ d'Euler sur X. On introduit la catégorie $\mathcal{C}$ dont les objets sont des $\mathcal{B}$-modules gradués de type fini T tels que $\dim_{\mathbb{C}} \mathbb{C}[\theta]u < \infty\ \forall u \in T$. Notons $\mathcal{C}' \subset \mathcal{C}$ la sous catégorie des $\mathcal{B}$-modules engendrés par les sections homogènes de degré entier (i.e. sections annulées par une puissance de $(\theta - p)$ avec p entier). Soit $\mathcal{C}_0 \subset \mathcal{C}'$ la sous catégorie des modules portés par l'origine $\{0\}$. Posons $\mathcal{C}'' := \mathcal{C}'/\mathcal{C}_0$ la catégorie quotient correspondante. On a le théorème :

Théorème 1. *Les catégories* $\mathrm{Mod}^{\mathrm{rh}}_{\tilde{\Lambda}}(\mathcal{D}_{\mathbb{CP}^{n-1}})$ *et* C'' *sont équivalentes.*

Nous établissons le résultat principal en décrivant les objets de la catégorie quotient $\mathcal{C}''$ en termes de diagrammes finis d'applications linéaires (cf. 4).

1. Introduction

One of the central problems in Algebraic Analysis concerns the complete description of holonomic systems with regular singularities. Several authors have taken an interest in it, notably Boutet de Monvel [1] who gave a very elegant presentation of regular holonomic $\mathcal{D}$-modules in one variable using pairs of finite dimensional $\mathbb{C}$-vector spaces related by certain linear maps. Deligne [3] gave a combinatorial description of the category of perverse sheaves on $\mathbb{C}$ with respect to the stratification $\{0\}$, $\mathbb{C}\backslash\{0\}$. It uses a characterisation of constructible sheaves given in [4,5]. These results are the starting point of several works on perverse sheaves in higher dimension (see for example [9]). Denote $X = \mathbb{C}^n$ and $\widetilde{X}$ its associated projective space $\mathbb{CP}^{n-1}$. As usual $\mathcal{D}_{\widetilde{X}}$ (resp. $\mathcal{D}_X$) will refer to the sheaf of differential operators on $\widetilde{X}$ (resp. X). Let $\widetilde{X}$ be stratified with respect to $(\widetilde{X}\backslash\widetilde{Q})$, $\widetilde{Q}$ where $\widetilde{Q}$ is a hypersurface defined by a quadratic form on $\mathbb{CP}^{n-1}$. Our purpose is to classify regular holonomic $\mathcal{D}_{\mathbb{CP}^{n-1}}$-modules $\mathcal{M}$ with characteristic variety $\mathrm{char}(\mathcal{M})$ contained in the union of conormal bundles to the strata of $\mathbb{CP}^{n-1}$:

$$\mathrm{char}(\mathcal{M}) \subset \tilde{\Lambda} := T^*_{(\widetilde{X}\backslash\widetilde{Q})}\widetilde{X} \cup T^*_{\widetilde{Q}}\widetilde{X}, \tag{2}$$

(equivalently: those that admits an infinitesimal action of $PO(n)$). They form an Abelian category we shall denote by $\mathrm{Mod}^{\mathrm{rh}}_{\tilde{\Lambda}}(\mathcal{D}_{\mathbb{CP}^{n-1}})$. Denote $\overline{\mathcal{B}} := \Gamma(X, \mathcal{D}_X)^G$ the Weyl algebra on X of G-invariant differential operators. We denote $\mathcal{B}$ the quotient of $\overline{\mathcal{B}}$ by $\mathcal{J} \subset \overline{\mathcal{B}}$ the ideal of operators vanishing on G-invariant functions. Let us introduce the category $\mathcal{C}$ consisting of graded $\mathcal{B}$-modules of finite type T such that $\dim_{\mathbb{C}} \mathbb{C}[\theta]u < \infty\ \forall u \in T$. Denote $\mathcal{C}' \subset \mathcal{C}$ the subcategory of graded $\mathcal{B}$-modules generated by 'homogeneous sections of integral degree'. Next, denote $\mathcal{C}_0 \subset \mathcal{C}'$ the subcategory consisting of modules with support at the origin $\{0\}$. Now, put $\mathcal{C}'' := \mathcal{C}'/\mathcal{C}_0$ the associated quotient category.

Theorem 1. *The categories* $\mathrm{Mod}^{\mathrm{rh}}_{\tilde{\Lambda}}(\mathcal{D}_{\mathbb{P}(X)})$ *and* $\mathcal{C}''$ *are equivalent i.e.*

$$\mathrm{Mod}^{\mathrm{rh}}_{\tilde{\Lambda}}(\mathcal{D}_{\mathbb{CP}^{n-1}}) \xrightarrow{\sim} \mathcal{C}''. \tag{3}$$

Finally, we establish the main result by encoding the objects in the quotient category $\mathcal{C}'' := \mathcal{C}'/\mathcal{C}_0$ by means of finite diagrams of linear maps (see Section 4).

2. Review on $\mathcal{D}_{\mathbb{C}^n}$-modules attached to a quadratic cone

First of all we refer the reader to [2,6–8] for notions on analytic $\mathcal{D}$-modules. Let us recall that if X denotes a complex manifold, a $\mathcal{D}_X$-module $\mathcal{M}$ is said to be holonomic if it is coherent and its characteristic variety $\mathrm{char}(\mathcal{M})$ is Lagrangian. Equivalently the characteristic variety is of dimension equal to $\dim X$ since it is always involutive. The holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is said to be regular holonomic (see [8, Corollary 5.1.11.]) if there exists a global filtration on $\mathcal{M}$ such that the annihilator of $\mathrm{gr}\,\mathcal{M}$ is a radical ideal in $\mathrm{gr}\,\mathcal{D}_X$. From now on $X = \mathbb{C}^n$, denote by q a nondegenerate quadratic form on X, $Q : q = 0$ (resp. $\widetilde{Q}$) the quadratic cone in $X = \mathbb{C}^n$ (resp. $\tilde{X} = \mathbb{CP}^{n-1}$). Then X (resp. $\widetilde{X} = \mathbb{CP}^{n-1}$) is stratified with respect to smooth complex spaces $X\backslash Q$, $Q\backslash\{0\}$, $\{0\}$ (resp. $(\widetilde{X}\backslash\widetilde{Q})$, $\widetilde{Q}$). Let $\Lambda := T^*_{(X\backslash Q)}X \cup T^*_{(Q\backslash\{0\})}X \cup T^*_{\{0\}}X$ (resp. $\tilde{\Lambda} := T^*_{(\widetilde{X}\backslash\widetilde{Q})}\widetilde{X} \cup T^*_{\widetilde{Q}}\widetilde{X}$) be the union of conormal bundles to these strata.

Denote by $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_X)$ (resp. $\mathrm{Mod}^{\mathrm{rh}}_{\tilde{\Lambda}}(\mathcal{D}_{\mathbb{CP}^{n-1}})$) the category whose objects are regular holonomic $\mathcal{D}_X$ (resp. $\mathcal{D}_{\tilde{X}}$)-modules with characteristic variety contained in Λ (resp. $\tilde{\Lambda}$).

Next, let $\mathcal{W}$ be the Weyl algebra on X. Denote by $G := \mathrm{SO}(q)$ the group of rotations and $\overline{\mathcal{B}} := \Gamma(X, \mathcal{D}_X)^G \subset \mathcal{W}$ the subalgebra of G -invariant differential operators. For any $x = (x_j) \in X$, denote by $q := q(x)$, $\Delta = \sum_{i,j=1}^{n} (\frac{\partial^2 q}{\partial x_i \partial x_j})^{-1} \frac{\partial^2}{\partial x_i \partial x_j}$ the Laplacian associated to the quadratic form q and $\theta = \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}$ the Euler vector field. Consider $\mathcal{J} := \overline{\mathcal{B}}(q\Delta - \theta(\theta + n - 2))\overline{\mathcal{B}} \subset \overline{\mathcal{B}}$ the two sided ideal (generated by $q\Delta - \theta(\theta + n - 2)$) of operators in $\overline{\mathcal{B}}$ vanishing on G-invariant functions. Put $\mathcal{B} := \overline{\mathcal{B}}/\mathcal{J}$. We recall (see [11, Proposition 2.1, p. 232]) the following:

Proposition 2. *The quotient algebra $\mathcal{B}$ is generated over $\mathbb{C}$ by q, Δ, θ such that*

$$[\theta, q] = 2q, \qquad [\theta, \Delta] = -2\Delta, \qquad [\Delta, q] = 4\theta + 2n.$$

Now, let us denote by $\mathcal{C}$ the category consisting of graded $\mathcal{B}$-module of finite type T such that $\dim_{\mathbb{C}} \mathbb{C}[\theta]u < \infty$ $\forall u \in T$.

For $n \neq 3$, we recall ([11, Theorem 3.9, p. 240]) the following result which will be effectively used in the next section.

Theorem 3. *For $n \neq 3$, the categories $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_{\mathbb{C}^n})$ and $\mathcal{C}$ are equivalent.*

For $n = 3$ (Q is not simply connected), $\forall (x, y, z) \in \mathbb{C}^3$ let $Q : xy = z^2$ and $i : (u, v) \mapsto (u^2, v^2, uv)$ a proper morphism from $\mathbb{C}^2$ to Q. Consider E the $\mathcal{D}_{\mathbb{C}^3}$-module supported by Q and generated by $f = i_+(u)$, $g = i_+(v)$ satisfying relations described in [11, Proposition 4.1, p. 243]. Then from [11, Theorem 4.2, p. 244] we have the following:

Theorem 4. *For $n = 3$, $\mathcal{M}$ in $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_{\mathbb{C}^3})$ decomposes as a sum of an object $\mathcal{N}$ in $\mathcal{C}$ and a copy of E i.e. $(\oplus_i^p E^i) \bigoplus \mathcal{N}$ with $\mathcal{N} \in \mathcal{C}$.*

3. Description of the inverse image

This section deals with the study of the inverse image, by the canonical projection $\pi : \mathbb{C}^n \backslash \{0\} \to \mathbb{CP}^{n-1}$, of a regular holonomic $\mathcal{D}_{\mathbb{CP}^{n-1}}$-module.

Definition 5. Let $\mathcal{N}$ be a $\mathcal{D}_X$-module. A section s in $\mathcal{N}$ is said to be homogeneous of integral degree $p \in \mathbb{Z}$, if there exists $j \in \mathbb{N}$ such that $(\theta - p)^j s = 0$.

As in the introduction, denote by $\mathcal{C}' \subset \mathcal{C}$ the subcategory consisting of graded $\mathcal{B}$-modules of finite type generated by homogeneous sections of integral degree. Denote by $\mathcal{C}_0 \subset \mathcal{C}'$ the subcategory consisting of modules supported by the origin $\{0\}$. We consider $\mathcal{C}'' := \mathcal{C}'/\mathcal{C}_0$ the corresponding quotient category. Let $\mathcal{M}$ be an object in $\mathrm{Mod}^{\mathrm{rh}}_{\tilde{\Lambda}}(\mathcal{D}_{\mathbb{CP}^{n-1}})$. In this section, the focus of our discussion will be the description of the inverse image of $\mathcal{M}$ by the canonical projection $\pi : \mathbb{C}^n \backslash \{0\} \to \mathbb{CP}^{n-1}$. It is a regular holonomic $\mathcal{D}_{X\backslash\{0\}}$-module (see [8, Corollary 5.4.8.]). We show that $\pi^+(\mathcal{M})$ is an object in the quotient category $\mathcal{C}''$. To do so let us first recall that $\mathcal{M}$ has a good filtration $\mathcal{M} = \bigcup_{k \in \mathbb{Z}} \mathcal{M}_k$ (see [8, Corollary 5.1.11 and A.5]). By using the Cartan Theorem A (see [12, Lemme 7]) we can see that for a large enough integer $d \in \mathbb{Z}^+$, the module $\mathcal{M}_k \otimes_{\mathcal{O}_{\mathbb{CP}^{n-1}}} \mathcal{O}(d)$ is generated over $\mathcal{O}_{\mathbb{CP}^{n-1}}$ by its global sections and $H^j(\mathbb{CP}^{n-1}, \mathcal{M}_k \otimes \mathcal{O}(d)) = 0$ for $j > 0$ (see. Cartan Theorem B in [12, Lemme 8]). Next $\mathcal{M} \otimes \mathcal{O}(d)$ is a $\mathcal{D}(d)$-module (with $\mathcal{D}(d) := \mathcal{O}(d) \otimes \mathcal{D}_{\mathbb{CP}^{n-1}} \otimes \mathcal{O}(-d)$). Then the sections in $\pi^*(\mathcal{M} \otimes \mathcal{O}(d))$ give the homogeneous sections of integral degree d in $\pi^+(\mathcal{M})$. Thus the regular holonomic $\mathcal{D}_{X\backslash\{0\}}$-module $\pi^+(\mathcal{M})$ is generated by homogeneous sections of integral degree $p \in \mathbb{Z}$. In addition these homogeneous generators are invariant by rotation.

Theorem 6. *$\pi^+(\mathcal{M})$ is generated over $\mathcal{D}_{X\backslash\{0\}}$ by a finite number of global homogeneous sections of "integral degree" and G-invariant i.e.*

$$\pi^+(\mathcal{M}) := \mathcal{D}_{X\backslash\{0\}} \left\{ s_1, \ldots, s_k \in \Gamma\big(X\backslash\{0\}, \pi^+(\mathcal{M})\big)^G \cap \left[\bigcup_{p \in \mathbb{Z}} \mathrm{Ker}(\theta - p) \right] \right\}.$$

3.1. Extension

This subsection consists in the extension of the inverse image $\pi^+(\mathcal{M})$. In other word, we see that there exists a regular holonomic $\mathcal{D}_X$-module $\mathcal{N}$ whose restriction on $X\setminus\{0\}$ is isomorphic to $\pi^+(\mathcal{M})$. Note $i:\mathbb{C}^n\setminus\{0\}\hookrightarrow\mathbb{C}^n$ the open embedding. We consider $\mathcal{N}:=i_+(\pi^+(\mathcal{M}))$ the direct image of $\pi^+(\mathcal{M})$ by the inclusion i. It is a regular holonomic $\mathcal{D}_X$-module (see [8, Theorem 6.2.1]) in $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_X)$ which extends $\pi^+(\mathcal{M})$ (see [10, Proposition 2.3]). Next, we distinguish two cases:

For $n\neq 3$, we can see by Theorem 3 that $i_+(\pi^+(\mathcal{M}))$ corresponds to an object we denote $\Psi(\mathcal{M})$ in the category $\mathcal{C}$. Therefore by Theorem 6, it turns out that $\pi^+(\mathcal{M})$ corresponds to an object $\tilde{\Psi}(\mathcal{M})$ in the quotient category $\mathcal{C}''$.

For $n=3$, since any conic $\widetilde{Q}$ in $\mathbb{CP}^2$ is simply connected (because $\widetilde{Q}$ is isomorphic to $\mathbb{CP}^1$) then in the lights of the results of the previous Section 2 $\pi^+(\mathcal{M})$ is associated to an object in the quotient category $\mathcal{C}''$.

Proposition 7. *For $\mathcal{M}$ in $\mathrm{Mod}^{\mathrm{rh}}_{\tilde{\Lambda}}(\mathcal{D}_{\mathbb{CP}^{n-1}})$, the inverse image $\pi^+(\mathcal{M})$ corresponds to an object $\tilde{\Psi}(\mathcal{M})$ in the quotient category $\mathcal{C}''$.*

3.2. Equivalence of categories

If $\widetilde{\mathcal{N}}$ is an object in the quotient category $\mathcal{C}''$, we associate to it the module $\tilde{\Phi}(\widetilde{\mathcal{N}})=\pi_+i^+(\mathcal{D}_X\otimes_{\mathcal{B}}\widetilde{\mathcal{N}})$ in the category $\mathrm{Mod}^{\mathrm{rh}}_{\tilde{\Lambda}}(\mathcal{D}_{\mathbb{CP}^{n-1}})$. Conversely, for any object $\mathcal{M}$ in $\mathrm{Mod}^{\mathrm{rh}}_{\tilde{\Lambda}}(\mathcal{D}_{\mathbb{CP}^{n-1}})$, its inverse image by π corresponds to an object $\tilde{\Psi}(\mathcal{M})$ in the quotient category $\mathcal{C}''$ (see Proposition 7). Consequently we obtain two functors $\tilde{\Psi}:\mathrm{Mod}^{\mathrm{rh}}_{\tilde{\Lambda}}(\mathcal{D}_{\mathbb{CP}^{n-1}})\to\mathcal{C}''$ (resp. $\tilde{\Phi}:\mathcal{C}''\to\mathrm{Mod}^{\mathrm{rh}}_{\tilde{\Lambda}}(\mathcal{D}_{\mathbb{CP}^{n-1}})$).

Theorem 8. *The functor $\tilde{\Psi}$ (resp. $\tilde{\Phi}$) establishes an equivalence of categories between the category $\mathrm{Mod}^{\mathrm{rh}}_{\tilde{\Lambda}}(\mathcal{D}_{\mathbb{CP}^{n-1}})$ and the quotient category $\mathcal{C}''$.*

4. Main result

We should remark that the objects in the quotient category $\mathcal{C}'':=\mathcal{C}'/\mathcal{C}_0$ can be understood by means of finite diagrams of linear maps. This section consists in the classification of such diagrams. Actually, a graded $\mathcal{B}$-module T in $\mathcal{C}'$ defines an infinite diagram consisting of finite dimensional complex vector spaces T_p (with $(\theta-p)$ being nilpotent on each T_p, $p\in\mathbb{Z}$) and linear maps between them deduced from the action of θ, q, Δ:

$$\cdots\rightleftarrows T_p\underset{\Delta}{\overset{q}{\rightleftarrows}}T_{p+2}\rightleftarrows\cdots \tag{4}$$

satisfying the relations of Proposition 2 and the following $(\theta-p)T_p\subset T_p$,

$$\theta=\frac{1}{4}\big([\Delta,q]-2n\big)\quad\text{and}\quad q\Delta=\theta(\theta+n-2),\quad \Delta q=(\theta+2)(\theta+n)\quad\text{on } T_p. \tag{5}$$

Let us give some examples of such graded $\mathcal{B}$-modules (in particular those with support in $\{0\}$ and their associated diagrams).

4.1. Examples

First, for any module in $\mathcal{C}_0$ supported by $\{0\}$, let us describe the corresponding diagram.

Example 9. The module $\mathcal{B}_{\{0\}|X}$ is generated by an element e_{-n} such that $\theta e_{-n}=-ne_{-n}$ and $qe_{-n}=0$. Then its associated graded $\mathcal{B}$-module T has a basis (e_m) where $m=-n-2k$ $(k\in\mathbb{N})$ such that $qe_{-n}=0$ and satisfying the following system:

$$S_0=\begin{cases}\theta e_m=me_m\quad (m=-n-2k,\,k\in\mathbb{N}),\\ \Delta e_m=e_{m-2},\\ qe_m=(m+2)(m+n)e_{m+2}.\end{cases} \tag{6}$$

Since $qe_{-n}=0$ (i.e. $qT_{-n}=0$), the arrows at the right of T_{-n} in the diagram vanish.

Second, we describe $\mathcal{B}$-modules associated to $\mathcal{O}_X$ and $\mathcal{O}_X(\frac{1}{q})/\mathcal{O}_X$.

Example 10. The module $\mathcal{O}_X$ is generated by an element $e_0 = 1_X$ such that $\theta e_0 = 0$ and $\Delta e_0 = 0$. Then its associated graded $\mathcal{B}$-module T has a basis (e_m) where $m = 2k$ $(k \in \mathbb{N})$ such that $\Delta e_0 = 0$ and satisfying the following system:

$$S_1 = \begin{cases} \theta e_m = m e_m \quad (m = 2k,\ k \in \mathbb{N}), \\ q e_m = e_{m+2}, \\ \Delta e_m = m(m+n-2)e_{m-2}. \end{cases} \tag{7}$$

Example 11. The module $\mathcal{O}_X(\frac{1}{q})/\mathcal{O}_X$ is generated by an element $e_{-2} = 1/q \bmod \mathcal{O}_X$ such that $\theta e_{-2} = -2e_{-2}$ and $qe_{-2} = e_0$. Then its associated graded $\mathcal{B}$-module T has a basis (e_m) with m an even integer $(m = -2k,\ k \in \mathbb{N})$ satisfying the following system:

$$S_2 = \begin{cases} \theta e_m = m e_m \quad (m = -2k,\ k \in \mathbb{N}), \\ \Delta e_m = e_{m-2}, \\ q e_m = (m+2)(m+n)e_{m+2}. \end{cases} \tag{8}$$

4.2. Classification of diagrams modulo $\mathcal{C}_0$

Now, any object in the quotient category $\mathcal{C}''$is a diagram $\widetilde{T} = T$ **modulo** $\mathcal{C}_0$ $(T \in \mathcal{C}')$

$$\cdots \rightleftarrows T_p \underset{\Delta}{\overset{q}{\rightleftarrows}} T_{p+2} \rightleftarrows \cdots \ \textbf{modulo}\ \mathcal{C}_0, \quad p \in \mathbb{Z}, \tag{9}$$

satisfying the previous relations (5). Such a diagram is completely determined by a finite subset of objects and arrows. Indeed, we distinguish two cases:

(a) *If n is odd and $p \equiv 0 \bmod 2\mathbb{Z}$ (resp. $p \equiv n \bmod 2\mathbb{Z}$)*, then $\widetilde{T}$ is completely determined by a diagram with 2 elements

$$T_{-2} \underset{\Delta}{\overset{q}{\rightleftarrows}} T_0 \ \textbf{modulo}\ \mathcal{C}_0 \tag{10}$$

$$(\text{resp. } T_{-n} \underset{\Delta}{\overset{q}{\rightleftarrows}} T_{-n+2} \ \textbf{modulo}\ \mathcal{C}_0). \tag{11}$$

In the other degrees q or Δ are bijective. Indeed, we have $T_0 \simeq q^k T_0 \simeq T_{2k}$ and $T_{-2} \simeq \Delta^k T_{-2} \simeq T_{-2-2k}$ $(k \in \mathbb{N})$ thanks to the relations (5) (resp. $T_{-n+2} \simeq q^k T_{-n+2} \simeq T_{-n+2(k+1)}$ and $T_{-n} \simeq \Delta^k T_{-n} \simeq T_{-n-2k)}$). The operator $q\Delta$ (resp. Δq) on T_p has only one eigenvalue $p(p+n-2)$ (resp. $(p+2)(p+n)$). Then the equation $q\Delta = \theta(\theta+n-2)$ (resp. $\Delta q = (\theta+2)(\theta+n)$) has a unique solution θ of eigenvalue p if p is not a critical value. Here $p = 0, -2, -n$, $-n+2$ thus it is always the case.

(b) *If n is even and $p \equiv 0 \bmod 2\mathbb{Z}$*. Then $\widetilde{T}$ is completely determined by:

either a diagram with 3 elements

$$T_{-n} \underset{b}{\overset{a}{\rightleftarrows}} T_{-2} \underset{\Delta}{\overset{q}{\rightleftarrows}} T_0 \ \textbf{modulo}\ \mathcal{C}_0 \tag{12}$$

(with $a = q^{(n-2)/2}$, $b = \Delta^{(n-2)/2}$, $q\Delta = \theta(\theta+n-2)$, $\Delta q = (\theta+2)(\theta+n)$ and the nilpotent action of $\frac{1}{4}([\Delta, q] - 2n) - p$ on each T_p)

or $\widetilde{T}$ may also be determined by all the diagram

$$T_{-n} \underset{\Delta}{\overset{q}{\rightleftarrows}} T_{-n+2} \rightleftarrows \cdots \rightleftarrows T_0 \ \textbf{modulo}\ \mathcal{C}_0 \tag{13}$$

with the operator θ (which we cannot reconstitute from q, Δ on $T_{\frac{-n-2}{2}}$ if $\frac{-n-2}{2}$ is even $(n = 2 \bmod 4)$).

(c) *If $p \neq 0, n \bmod 2\mathbb{Z}$ (p integer)*, then q and Δ are bijective. Thus $\widetilde{T}$ is completely determined up to a isomorphism by one element T_p **modulo** $\mathcal{C}_0$ equipped with the nilpotent action of $(\frac{1}{4}([\Delta, q] - 2n) - p)$ on each T_p.

Finally we should note that there are 5 indecomposable such $\mathcal{D}_{\mathbb{CP}^{n-1}}$-modules that is the pull backs (from $\mathbb{C}^n \backslash \{0\}$) of $\mathcal{O}_X$, $\mathcal{O}_X[1/q]/\mathcal{O}_X$, $\mathcal{D}_X.q^{1/2}$, $\mathcal{O}_X[1/q]$, $\mathcal{D}_X.\log q/\mathcal{O}_X$ (the first 3 are irreducible). Moreover any $\mathcal{D}_{\mathbb{CP}^{n-1}}$-module of the type studied here is a finite direct sum of submodules isomorphic to one of these. This can be seen on our previous diagrams, knowing that two of our diagrams for $\mathbb{C}^n$ which coincide in degree $> -n$ give the same $\mathcal{D}$-module on the projective space- or more elementarily if one knows $\pi_1(\mathbb{CP}^n) = \pi_1(Q) = 0$, $\pi_1(\mathbb{CP}^n - Q) = \mathbb{Z}/2\mathbb{Z}$ if $n > 1$. We have the following:

Proposition 12. *A regular holonomic $\mathcal{D}_{\mathbb{CP}^{n-1}}$-module in $\mathrm{Mod}^{\mathrm{rh}}_{\tilde{\Lambda}}(\mathcal{D}_{\mathbb{CP}^{n-1}})$, is a direct sum of indecomposable submodules. These are isomorphic to one of the modules "pull backs" (from $\mathbb{C}^n\backslash\{0\}$) of $\mathcal{O}_X$, $\mathcal{O}_X[1/q]/\mathcal{O}_X$, $\mathcal{D}_X.q^{1/2}$, $\mathcal{O}_X[1/q]$, $\mathcal{D}_X.\log q/\mathcal{O}_X$.*

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