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Probability Theory

The limiting spectral measure of the Generalised Inverse Gaussian random matrix model

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Abstract

In this Note, we provide a complete description of the limiting spectral measure of the Generalised Inverse Gaussian (GIG) random matrix model. The overall strategy relies on the large deviation theorem for the empirical measure of general continuous Coulomb gas. This limit is given as the extremal measure of a weighted logarithmic energy problem which may be explicitly solved here. Furthermore, we prove the almost sure convergence of the largest and smallest eigenvalues of the GIG model towards respectively the right and left-endpoints of the extremal compact support. ***To cite this article: D. Féral, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

La mesure spectrale limite du modèle de matrices aléatoires de la loi Gaussienne Inverse Généralisée. Dans cette Note, nous explicitons la mesure spectrale limite du modèle de matrices aléatoires soumises à la loi Gaussienne Inverse Généralisée. La stratégie sous-jacente repose sur un théorème de grandes déviations établi pour la mesure spectrale de certains gaz de Coulomb plus généraux. Cette limite apparaît comme la mesure extrémale d'un problème d'énergie logarithmique avec poids qui peut être complètement résolu ici. De plus, nous prouvons la convergence presque sûre de la plus grande (resp. petite) valeur propre vers le bord droit (resp. gauche) du support compact extrémal. ***Pour citer cet article : D. Féral, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

The Generalised Inverse Gaussian (GIG) random matrix model generalises the famous Wishart model (cf. [2] and [4]) and is of interest in Statistics. It may be described as follows. Given an integer $N \geqslant 1$, we denote by $\mathcal{H}_N^+(\mathbb{R})$ the space of the $N \times N$ Hermitian positive definite matrices. For A_N, B_N in $\mathcal{H}_N^+(\mathbb{R})$ and λ_N in $\mathbb{R}$, a random matrix X_N of $\mathcal{H}_N^+(\mathbb{R})$ is said to have a GIG distribution if its law is given by

$$\mu_{\lambda_N, A_N, B_N}(\mathrm{d}X_N) = \frac{(\det X_N)^{\lambda_N - N}}{K_N(A_N, B_N, \lambda_N)} \exp\left[-\operatorname{Tr}\left(A_N X_N + B_N X_N^{-1}\right)\right] \mathbf{1}_{\mathcal{H}_N^+(\mathbb{R})}(X_N)\,\mathrm{d}X_N \tag{1}$$

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where det is the determinant, Tr the trace, $\mathrm{d}X_N$ the Lebesgue measure on $\mathcal{H}_N^+(\mathbb{R})$ and $K_N(A_N, B_N, \lambda_N)$ the finite normalisation constant. For $B_N = 0$ and $\lambda_N > 0$, this is the well-known Wishart distribution with covariance matrix A_N (see [1]). Denote by $x_1 \leqslant \cdots \leqslant x_N$ the N eigenvalues of X_N. Assume $A_N = \alpha_N I_N$ and $B_N = \beta_N I_N$ with $\alpha_N, \beta_N > 0$ (I_N is the identity matrix). By (1) and the Jacobian formula, the joint density of the eigenvalues on the Weyl chamber $E = \{x \in \mathbb{R}^N;\ 0 < x_1 < \cdots < x_N\}$ may be shown (cf. [10]) to be given by the Coulomb gas representation

$$GIG(\lambda_N, \alpha_N, \beta_N)(x) := \frac{1}{Z_N}\left|\Delta_N(x)\right|^2 \exp\left(-N\sum_{i=1}^{N} V_N(x_i)\right)1_E(\mathrm{d}x) \tag{2}$$

where $\Delta_N(x) = \prod_{i<j}(x_i - x_j)$ is the Vandermonde determinant, Z_N the normalisation constant and

$$V_N(x) = \left(1 - \frac{\lambda_N}{N}\right)\ln x + \frac{\alpha_N}{N}x + \frac{\beta_N}{Nx}. \tag{3}$$

By symmetry under permutations, the joint density (2) may be extended to the whole of $(\mathbb{R}^{+*})^N$. As

$$\frac{\lambda_N}{N} \to \lambda \in \mathbb{R}, \quad \frac{\alpha_N}{N} \to \alpha > 0, \quad \frac{\beta_N}{N} \to \beta > 0, \quad \text{when } N \to \infty, \tag{4}$$

note that

$$V_N(x) \to V(x) = (1-\lambda)\ln x + \alpha x + \frac{\beta}{x}, \quad \text{uniformly on any compact subset of } \mathbb{R}^{+*}. \tag{5}$$

In this Note, we shall be interested in the limiting spectral distribution on the eigenvalues $\hat{\mu}_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ as the size $N \to \infty$. We further investigate the extremal eigenvalues $x_{\max} := \max_i x_i$ and $x_{\min} := \min_i x_i$ (we suppress dependence upon N). We restrict ourselves to $\beta > 0$. When $\beta = 0$, $\lambda > 0$ and $\alpha = 1$, the model is the classical complex Wishart Ensemble considered in [8] and [6].

Continuous Coulomb gases of type (2) with more general potentials V_N and V have been investigated in the literature (see [3,7,6] and references therein). In particular, in the work [6], a general framework including GIG model has been developed for which a Large Deviation Principle (LDP) and the almost sure convergence for both $\hat{\mu}_N$ and $x_{\max}$ have been established. In the LDP formulation, the limiting spectral measure μ_V is usually described as the extremal measure of a weighted logarithmic energy problem. As the main conclusion here, we show how μ_V may be explicitly described for the GIG, giving rise to precise limiting statements for both $\hat{\mu}_N$ and the extremal eigenvalues. Below, $\mathcal{M}(\mathbb{R}^{+*})$ denotes the space of the probability measures on $\mathbb{R}^{+*}$ equipped with the classical Lévy distance.

Theorem 1. *Under the $GIG(\lambda_N, \alpha_N, \beta_N)$ distribution, $(\hat{\mu}_N)_N$ satisfies on $\mathcal{M}(\mathbb{R}^{+*})$ a LDP with speed N^2 and good rate function (GRF)*

$$\forall \mu \in \mathcal{M}(\mathbb{R}^{+*}), \quad I_V(\mu) = \iint_{\mathbb{R}^{+*}\times\mathbb{R}^{+*}} \log|x-y|^{-1}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) + \int_{\mathbb{R}^{+*}} V(x)\,\mathrm{d}\mu(x) - F_V \tag{6}$$

where F_V is a finite (explicit) constant. Moreover, $(\hat{\mu}_N)$ converges almost surely (a.s.) to the deterministic extremal probability μ_V which minimizes I_V, which is compactly supported on $\mathbb{R}^{+*}$ and given by

$$\frac{\mathrm{d}\mu_V}{\mathrm{d}x} = \frac{1}{2\pi}\sqrt{(x-a)(b-x)} \times \left[\frac{\alpha}{x} + \frac{\beta}{\sqrt{ab}\,x^2}\right]1_{[\mathrm{a},\mathrm{b}]}(x) \tag{7}$$

where $0 < a < b$ are solution of

$$1 - \lambda + \alpha\sqrt{ab} - \beta\frac{a+b}{2ab} = 0 \quad \textit{and} \quad 1 + \lambda + \frac{\beta}{\sqrt{ab}} - \alpha\frac{a+b}{2} = 0. \tag{8}$$

The next theorem derived from Section 4 of [6] deals with the largest eigenvalue $x_{\max}$.

Theorem 2. *Under the $GIG(\lambda_N, \alpha_N, \beta_N)$, $x_{\max}$ converges a.s. to b and satisfies on $\mathbb{R}^{+*}$a LDP with speed N and GRF*

$$I^*_{\alpha,\beta,a,b}(t) = \begin{cases} \displaystyle\int_b^t \frac{1}{2x}\left(\alpha + \frac{\beta}{x\sqrt{ab}}\right)\sqrt{(x-a)(x-b)}\,\mathrm{d}x & \textit{if } t \geqslant b,\\ +\infty & \textit{otherwise.}\end{cases} \tag{9}$$

Some specific symmetry properties of the GIG model actually leads to an analogous conclusion for $x_{\min}$.

Theorem 3. *Under the GIG($\lambda_N, \alpha_N, \beta_N$), $x_{\min}$ tends a.s. to a, satisfies a LDP with speed N and GRF*

$$I_V^{**}(t) = \begin{cases} I^*_{\beta,\alpha,\frac{1}{b},\frac{1}{a}}(t) & \text{if } t \leqslant a, \\ +\infty & \text{otherwise.} \end{cases} \tag{10}$$

Notice that with minor modifications, the results above also hold in the real setting. As already mentioned, general versions of Theorems 1 and 2 have been obtained in [6]. In the next section, we outline the proof of the explicit expression of μ_V, explain the rate function $I^*_{\alpha,\beta,a,b}$ and justify Theorem 3.

2. Sketch of proofs

The existence and unicity of μ_V follow from the theory of 'the energy problem' developed in the book [11]. Since V in (5) is regular enough (see [6]), μ_V has a density Φ_V and is compactly supported on $\mathbb{R}^{+*}$.

Let us first determine the compact support $[a, b]$. By Theorem IV.1.11 of [11], its edges a and b satisfy

$$\frac{1}{\pi}\int_a^b \frac{V'(x)}{\sqrt{(b-x)(x-a)}}\,\mathrm{d}x = 0 \quad \text{and} \quad \frac{1}{\pi}\int_a^b \frac{xV'(x)}{\sqrt{(b-x)(x-a)}}\,\mathrm{d}x = 2. \tag{11}$$

Since $V'(x) = \frac{1-\lambda}{x} + \alpha - \frac{\beta}{x^2}$, to obtain (8), we shall compute several integrals and make use of some arguments of complex analysis (cf. [9] and Section V of [5]). For example, we need to establish that

$$\frac{1}{\pi}\int_a^b \frac{1}{x^2}\frac{1}{\sqrt{(x-a)(b-x)}}\,\mathrm{d}x = \frac{a+b}{2ab\sqrt{ab}}. \tag{12}$$

Our approach is first to define, for all $z \in \mathbb{C} \setminus [a, b]$, $\sqrt{(z-a)(z-b)} = \exp\{\frac{1}{2}\mathrm{Log}[(z-a)(z-b)]\}$ where Log is the principal determination of the logarithm. Let $K_{\epsilon,R} = \{z \in \mathbb{C}\colon\ d(z, [a, b]) \geqslant \epsilon,\ |z| \leqslant R\}$ with $\epsilon, R > 0$, and call $\gamma_{\epsilon,R}$ its boundary. By the Residue Theorem and the Cauchy formula,

$$\frac{-1}{2\mathrm{i}\pi}\int_{\gamma_{\epsilon,R}} \frac{1}{\sqrt{(z-a)(z-b)}}\frac{\mathrm{d}z}{z^2} = \frac{a+b}{2ab\sqrt{ab}} \quad (\text{under } 0 < \epsilon < a \text{ and } R > b + \epsilon). \tag{13}$$

Then, letting $R \to \infty$ and $\epsilon \to 0$, one easily shows that the R.H.S. of (13) tends to that of (12).

From Theorem IV.3.1 of [11], the density $\Phi_V = \frac{\mathrm{d}\mu_V}{\mathrm{d}x}$ is given by (where PV denotes the principal value)

$$\Phi_V(x) = \frac{1}{2\pi}\sqrt{(x-a)(b-x)}\,PV\left(\frac{1}{\pi}\int_a^b \frac{V'(t)}{\sqrt{(t-a)(b-t)}}\frac{\mathrm{d}t}{t-x}\right).$$

Once one has noticed that, for all $a < x < b$, $\frac{1}{\pi}\int_a^b \frac{1}{t-x}\frac{\mathrm{d}t}{\sqrt{(t-a)(b-t)}} = 0$ (this is a consequence of Theorem I.1.3 and Example I.3.5 of [11]), (7) can be derived using again the Residue method.

To prove Theorem 2, observe first that the GIG model fulfils the assumptions of Section 4 of [6] since

$$g(x) = \int_a^b \log|x-t|^{-1}\Phi_V(t)\,\mathrm{d}t + \frac{1}{2}V(x) + \frac{1}{2}\int_a^b V(t)\Phi_V(t)\,\mathrm{d}t$$

is a non-decreasing function on $]b, +\infty[$. Indeed, $g'(x) = m_V(x) + \frac{1}{2}V'(x)$ where $m_V(x) = \int_a^b \frac{1}{t-x}\Phi_V(t)\,\mathrm{d}t$ is the Stieltjes transform of μ_V. It is not hard to show (once more using the Residue Theorem) that

$$\forall x \in \mathbb{C} \setminus [a, b], \quad m_V(x) = -\frac{1}{2}V'(x) + \frac{1}{2x}\left(\alpha + \frac{\beta}{x\sqrt{ab}}\right)\sqrt{(x-a)(x-b)}.$$

This leads for all $x > b$ to $g'(x) = \frac{1}{2x}(\alpha + \frac{\beta}{x\sqrt{ab}})\sqrt{(x-a)(x-b)}$ which is non-negative. Now, the announced formula for $I^*_{\alpha,\beta,a,b}$ is a consequence of the fact (see [6]) that for all $t \geqslant b$, $I^*_{\alpha,\beta,a,b}(t) = \int_b^t g'(x)\,\mathrm{d}x$.

Finally, Theorem 3 is deduced from Theorem 2 thanks to the following interesting property of the GIG.

Lemma 4. *If X_N has the GIG$(\lambda_N, \alpha_N, \beta_N)$ distribution, then X_N^{-1} is of the GIG$(-\lambda_N, \beta_N, \alpha_N)$.*

This lemma follows from (1) (see [10] or [2]) (or also from (2)). At last, one needs only notice that the associated extremal support of the *GIG*$(-\lambda_N, \beta_N, \alpha_N)$ is $[\frac{1}{b}, \frac{1}{a}]$ if $[a, b]$ is that of the *GIG*$(\lambda_N, \alpha_N, \beta_N)$.

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