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Group Theory

Asymptotic aspects of Schreier graphs and Hanoi Towers groups

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Abstract

We present relations between growth, growth of diameters and the rate of vanishing of the spectral gap in Schreier graphs of automaton groups. In particular, we introduce a series of examples, called Hanoi Towers groups since they model the well known Hanoi Towers Problem, that illustrate some of the possible types of behavior. ***To cite this article: R. Grigorchuk, Z. Šunić, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Aspects asymptotiques des graphes de Schreier et groupes des Tours de Hanoï. On montre quelques relations entre la croissance, la croissance des diamètres et la vitesse avec laquelle le trou spectral dans les graphes de Schreier des groupes automatiques tend vers zéro. En particulier, on introduit un certain nombre d'exemples, les groupes dits des Tours de Hanoï car ils donnent un modèle du célèbre problème des Tours de Hanoï, et qui illustrent des types possibles de comportement. ***Pour citer cet article : R. Grigorchuk, Z. Šunić, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Étant donnée une permutation $\pi \in \mathsf{S}_k$ on définit un automorphisme $a = a_\pi$ du k-arbre enraciné en posant $a = \pi\ (a_0, a_1, \ldots, a_{k-1})$, où a_i est l'automorphisme identité si i est dans le support de π et $a_i = a$ sinon. L'action de l'automorphisme $a_{(ij)}$ sur l'arbre est donnée (récursivement) par (1). Le *groupe des Tours de Hanoï sur k piquets*, $k \geqslant 3$, est le groupe $H^{(k)} = \langle \{a_{(ij)} \mid 0 \leqslant i < j \leqslant k-1\} \rangle$ des automorphismes du k-arbre engendré par les $a_{(ij)}$, $0 \leqslant i < j \leqslant k-1$. Le groupe $H^{(k)}$ doit son nom au fait qu'il modèle le problème bien connu des Tours de Hanoï sur k piquets (voir [6,14]). L'action de l'automorphisme $a_{(ij)}$ correspond à un déplacement entre les piquets i et j.

Théorème 0.1. *$H^{(3)}$ est contractant et régulièrement branché sur son groupe des commutateurs.*

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Soit $A=(S,X,\tau,\rho)$ un automate, $G=G(A)$ le groupe d'automate correspondant qui agit sur le k-arbre enraciné, soit ξ un rayon géodésique infini commençant à la racine, soit ξ_n le préfixe de ξ de longueur n, soit P_ξ le stabilisateur $\mathsf{St}_G(\xi)$ et soit P_n le stabilisateur $\mathsf{St}_G(\xi_n)$, pour $n=0,1,\ldots$. Dénotons par Γ_ξ (ou simplement par Γ) le graphe de Schreier $\Gamma=\Gamma(G,P_\xi,S)$ et par Γ_n le graphe de Schreier $\Gamma_n=\Gamma_n(G,P_n,S)$. Par exemple, dans le cas du groupe de Hanoï sur 3 piquets $H^{(3)}$, le graphe de Schreier Γ_3 est donné en Fig. 1 (les automorphismes $a_{(01)}$, $a_{(02)}$ et $a_{(12)}$ sont dénotés a, b et c, respectivement).

La croissance de Γ peut être exponentielle, intermédiaire ou polynomiale.

Théorème 0.2. *Pour le groupe des Tours de Hanoï $H^{(k)}$, $k\geqslant 4$, la croissance de $\Gamma_{000\ldots}$ est intermédiaire.*

Dénotons par $d(n)$ le diamètre du graphe Γ_n. La croissance de la fonction $d(n)$ est exponentielle, intermédiaire ou polynomiale.

Théorème 0.3. *Pour le groupe des Tours de Hanoï $H^{(k)}$, $k\geqslant 3$, $d(n)$ est asymptotiquement* $\mathrm{e}^{n^{\frac{1}{k-2}}}$.

Soit T_n la matrice d'adjacence de Γ_n, soit $M_n=\frac{1}{|S\cup S^{-1}|}T_n$ l'opérateur de Markov correspondant, soit T la matrice d'adjacence de Γ et soit M l'opérateur de Markov correspondant. Le *spectre* de Γ_n (ou de Γ) est, par définition, le spectre de M_n (ou de M). Soit $\delta(n)$ le *trou spectral* $1-\lambda(n)$, où $\lambda(n)$ est la plus grande valeur propre de Γ_n autre que 1. Le trou spectral peut être éloigné de 0 ($\{\Gamma_n\}$ forme une famille d'expanseurs) ou bien peut s'approcher de 0 de façon exponentielle, intermédiaire ou polynomiale.

Théorème 0.4. *Soit G le groupe des Tours de Hanoï sur 3 piquets $H^{(3)}$. Le spectre de Γ_n, en tant qu'ensemble, a $3\cdot 2^{n-1}-1$ éléments et est égal (quitte à changer l'échelle d'un facteur 3) à (2), où f est le polynôme $f(x)=x^2-x-3$. La multiplicité des 2^i valeurs propres dans $f^{-i}(0)$, pour $i=0,\ldots,n-1$, est a_{n-i}, et la multiplicité des 2^j valeurs propres dans $f^{-j}(-2)$, pour $j=0,\ldots,n-2$, est b_{n-j}, où $a_i=(3^{i-1}+3)/2$ et $b_j=(3^{j-1}-1)/2$, pour $i,j\geqslant 1$. Le spectre de $\Gamma_{000\ldots}$, en tant qu'ensemble, est égal (à changement d'échelle d'un facteur 3 près) à (3). Il consiste en l'ensemble des points isolés $I=\bigcup_{i=0}^{\infty}f^{-i}\{0\}$ et son ensemble de points d'adhérence J, qui est l'ensemble de Julia du polynôme f et est un ensemble de Cantor. La mesure spectrale KNS est discrète, concentrée en $\bigcup_{i=0}^{\infty}f^{-i}\{0,-2\}$, et la mesure des valeurs propres dans $f^{-i}\{0,-2\}$ est $1/(6\cdot 3^i)$, pour $i=0,1,\ldots$.*

1. Actions on rooted trees, automaton groups, and Hanoi Towers groups

The free monoid X^* of words over the alphabet $X=\{0,\ldots,k-1\}$ ordered by the prefix relation has a k-regular rooted tree structure in which the empty word is the root and the words of length n constitute the level n in the tree. The k children of the vertex u are the vertices ux, for $x=0,\ldots,k-1$. Denote this k-regular rooted tree by $\mathcal{T}$. Any automorphism g of $\mathcal{T}$ can be (uniquely) decomposed as $g=\pi_g\,(g_0,g_1,\ldots,g_{k-1})$, where $\pi_g\in\mathsf{S}_k$ is called the *root permutation* of g, and g_x, $x=0,\ldots,k-1$, are tree automorphisms, called the (first level) *sections* of g. The root permutation π_g and the sections g_i are determined uniquely by the relation $g(xw)=\pi_g(x)g_x(w)$, for all $x\in X$ and $w\in X^*$. The action of a tree automorphism can be extended to an isometric action on the boundary $\partial\mathcal{T}$ consisting of the infinite words over X. The space $\partial\mathcal{T}$ is a compact ultrametric space homeomorphic to a Cantor set.

For any permutation π in S_k define a k-ary tree automorphism $a=a_\pi$ by $a=\pi\,(a_0,a_1,\ldots,a_{k-1})$, where a_i is the identity automorphism if i is in the support of π and $a_i=a$ otherwise. The action of the automorphism $a_{(ij)}$ on $\mathcal{T}$ is given (recursively) by

$$a_{(ij)}(iw)=jw,\qquad a_{(ij)}(jw)=iw,\qquad a_{(ij)}(xw)=xa_{(ij)}(w),\quad \text{for } x\notin\{i,j\}. \tag{1}$$

Hanoi Towers group on k pegs, $k\geqslant 3$, is the group $H^{(k)}=\langle\{a_{(ij)}\mid 0\leqslant i<j\leqslant k-1\}\rangle$ of k-ary tree automorphisms generated by the automorphisms $a_{(ij)}$, $0\leqslant i<j\leqslant k-1$, corresponding to the transpositions in S_k. The group $H^{(k)}$ derives its name from the fact that it models the well known Hanoi Towers Problem on k pegs (see [6,14]). In this Problem, n disks of distinct sizes, enumerated $1,\ldots,n$ by their size, are placed on k pegs, denoted $0,\ldots,k-1$. Any configuration of disks is allowed as long as no disk is placed on top of a smaller disk. A legal move consists of moving the top disk from one peg to the top of another peg (as long as the new configuration is allowed). The n-disk

configurations are in bijective correspondence with the k^n words of length n over $X = \{0, \dots, k-1\}$. Namely, the word $x_1 \dots x_n$ over X corresponds to the unique configuration in which the disk i, $i = 1, \dots, n$, is placed on peg x_i. The action of the automorphism $a_{(ij)}$ corresponds to a move between the pegs i and j.

The group $H^{(k)}$, $k \geqslant 3$, is an example of an automaton group. In general, an *invertible automaton* is a quadruple $A = (S, X, \tau, \rho)$ in which S is a finite set of states, X a finite alphabet, $\tau : S \times X \to S$ a *transition function* and $\pi : S \times X \to X$ an *output function* such that, for each state $s \in S$, the restriction $\pi_s = \pi(s, \cdot) : X \to X$ is a permutation in S_X (see [11]). The states of A define recursively tree automorphisms by setting the permutation π_s to be the root permutation of s and the state $\tau(s, x)$ to be the section s_x of s at x. The group of tree automorphisms $G(A) = \langle s \mid s \in S \rangle$ generated by the automorphisms corresponding to the states of the invertible automaton A is called the *automaton group* of A. Invertible automata are often represented by diagrams such as the one on the left in Fig. 1 corresponding to $H^{(4)}$. Each state s is labeled by the permutation π_s and the labeled edges describe the transition function (if $\tau(s, x) = t$ then there is an edge labeled x connecting s to t).

An automaton group is *contracting* if the length (with respect to the generating set S) of each section g_i, $i = 0, \dots, k-1$, of g is shorter than the length of g, for all sufficiently long elements g. A spherically transitive (transitive on all levels) group of tree automorphisms G is *regularly branching* over its subgroup K if K is a normal subgroup of finite index in G such that $K \times \cdots \times K$ (k factors) is a normal subgroup of finite index in G that is geometrically contained in K (meaning that the k factors act independently on the corresponding k subtrees of $\mathcal{T}$ rooted at the vertices of the first level of $\mathcal{T}$; see [3]).

Theorem 1.1. *The Hanoi Towers group $H^{(3)}$ is contracting and regularly branching over its commutator.*

The groups $H^{(k)}$ act spherically transitively on $\mathcal{T}$, but are not contracting for $k \geqslant 4$.

2. Schreier graphs

Let A be an automaton, $G = G(A)$ the corresponding automaton group, ξ an infinite geodesic ray starting at the root of $\mathcal{T}$, ξ_n the prefix of ξ of length n, P_ξ the stabilizer $\mathsf{St}_G(\xi)$ and P_n the stabilizer $\mathsf{St}_G(\xi_n)$, $n = 0, 1, \dots$. Denote by Γ_ξ (or just Γ) the Schreier graph $\Gamma = \Gamma(G, P_\xi, S)$ and by Γ_n the Schreier graph $\Gamma_n = \Gamma_n(G, P_n, S)$, where S is the generating set defined by the states of $A = (S, X, \tau, \rho)$. We assume that the action of G on $\mathcal{T}$ is spherically transitive. Thus the graphs Γ_n are connected, have size k^n, $n = 0, 1, \dots$, and G is infinite. The graph Γ_n is indeed the graph of the action of G on level n in the tree and Γ is the graph of the action on the orbit of ξ in $\partial\mathcal{T}$. The sequence of graphs $\{\Gamma_n\}$ converges to the infinite graph Γ in the space of pointed graphs [12] (based at ξ_n, $n = 0, 1, \dots$, and ξ, respectively).

For example, in the case of the Hanoi group on 3 pegs $H^{(3)}$ the Schreier graph Γ_3 is given in Fig. 1 (the automorphisms $a_{(01)}$, $a_{(02)}$ and $a_{(12)}$ are denoted by a, b and c, respectively in the figure). The group $H^{(3)}$ has been independently constructed in [15] as a group whose limit space is the Sierpiński gasket.

All our asymptotic considerations are intended in the following sense. For two functions, we write $f \preceq g$ if there exists $C > 0$ such that $f(n) \leqslant g(Cn)$, for all $n \geqslant 0$, and $f \sim g$ if $f \preceq g$ and $g \preceq f$.

2.1. Growth

Denote by $\gamma(n)$ the growth function of Γ counting the number of vertices in Γ that are in the ball of radius n centered at ξ. The growth of Γ is either exponential ($\lim_{n\to\infty} \sqrt[n]{\gamma(n)} = c > 1$), intermediate ($\lim_{n\to\infty} \sqrt[n]{\gamma(n)} = 1$, $n^m \preceq \gamma(n)$, for all $m > 0$) or polynomial ($\lim_{n\to\infty} \sqrt[n]{\gamma(n)} = 1$, $\gamma(n) \preceq n^m$, for some $m > 0$). The growth is polynomial in any Schreier graph Γ associated to a contracting group [2]. For example, $\gamma(n) \sim n^{\log_2(3)}$ in case of $H^{(3)}$, and the growth is linear for the first group from [10] and the Erschler group [8]. The growth of Γ is exponential for the automata generating the lamplighter group [13] and Baumslag-Solitar solvable groups [4], the Bellaterra automaton [15] generating the free product of three cyclic groups of order 2, and the Aleshin automaton [1,18] generating the free group of rank 3.

Theorem 2.1. *For the Hanoi Towers group $H^{(k)}$, $k \geqslant 4$, the growth of $\Gamma_{000\dots}$ is intermediate. Moreover, $a^{(\log n)^{k-2}} \preceq \gamma(n) \preceq b^{(\log n)^{k-2}}$, for some constants $b > a > 1$.*

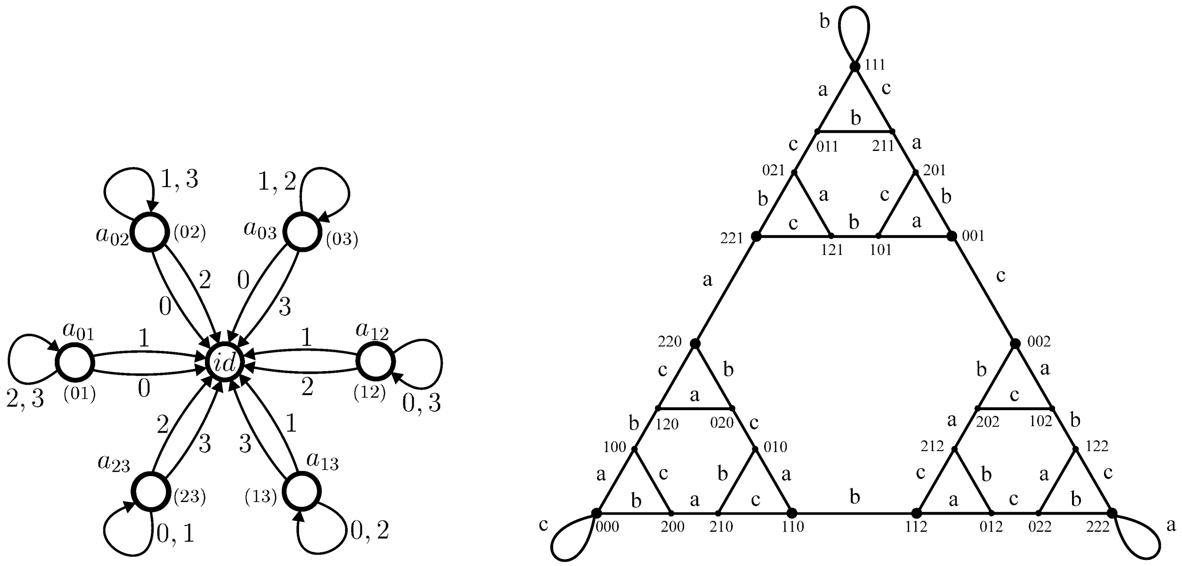

Fig. 1. The automaton generating $H^{(4)}$ and the Schreier graph of $H^{(3)}$ at level 3.
Fig. 1. L'automate engendrant $H^{(4)}$ et le graphe de Schreier de $H^{(3)}$ au niveau 3.

Thus the Schreier graph Γ is amenable, for all Hanoi Towers groups (another example providing Schreier graphs of intermediate growth is the group studied in [5,7]). It is not known if, for $k \geqslant 4$, $H^{(k)}$ is amenable or if it contains free subgroups of rank 2 ($H^{(3)}$ is amenable but not elementary amenable).

2.2. Diameters

Denote by $d(n)$ the diameter of the graph Γ_n. The growth of the diameter function $d(n)$ is either exponential, intermediate or polynomial. Since we assumed spherical transitivity of the action, we have that the Schreier graphs Γ_n are connected and $\gamma(d(n)) \geqslant k^n$. This implies that if either $d(n)$ or $\gamma(n)$ grows polynomially then the other one grows exponentially. Thus, contracting groups provide examples with exponential diameter growth. As a concrete example, it is well known that $d(n) = 2^n - 1$ for $H^{(3)}$. It can be shown that examples of polynomial (even linear) diameter growth are given by the realizations of the lamplighter group in [13] and by Baumslag-Solitar solvable groups $BS(1,n)$, $n \neq \pm 1$, in [4].

Theorem 2.2. *For the Hanoi Towers group $H^{(k)}$, $k \geqslant 3$, the diameter $d(n)$ is asymptotically* $\mathrm{e}^{n^{\frac{1}{k-2}}}$.

Thus the Hanoi Towers groups provide examples with intermediate diameter growth (for $k \geqslant 4$). The classical Hanoi Towers Problem asks for the number of steps needed to reach the configuration 1^n from 0^n. It can be shown that this distance is asymptotically equal (it is not equal!) to the diameter $d(n)$. The Frame–Stewart algorithm (see [14]) solves the Hanoi Towers Problem in $\sim \mathrm{e}^{n^{\frac{1}{k-2}}}$ steps and it follows from the work of Szegedy [16] that this is asymptotically optimal.

2.3. Spectra and spectral gap

Let T_n be the adjacency matrix of Γ_n and $M_n = \frac{1}{|S \cup S^{-1}|} T_n$ be the corresponding Markov operator (related to the simple random walk on Γ_n). Similarly, let T be the adjacency matrix of Γ and M the corresponding Markov operator. The *spectrum* of Γ_n (or Γ) is, by definition, the spectrum of M_n (or M). The regularity of Γ_n implies that 1 is in the spectrum of Γ_n. Let $\delta(n)$ be the *spectral gap* $1 - \lambda(n)$, where $\lambda(n)$ is the largest eigenvalue of Γ_n different from 1. The family $\{\Gamma_n\}$ is, by definition, a family of *expanders* if there is a constant $c > 0$ such that $\delta(n) \geqslant c$, for all n. If $\{\Gamma_n\}$ do not form a family of expanders then the spectral gap tends to 0 at a rate that is exponential ($\lim_{n\to\infty} \sqrt[n]{\delta(n)} = c < 1$), intermediate ($\lim_{n\to\infty}\sqrt[n]{\delta(n)}=1$, $\delta(n)\preceq n^{-m}$, for all $m>0$) or polynomial ($\lim_{n\to\infty}\sqrt[n]{\delta(n)}=1$, $n^{-m}\preceq\delta(n)$, for some $m>0$). Chung inequality tells us that, for a graph $\Gamma=(V,E)$, the diameter $d\leqslant\frac{\ln(|V|-1)}{-\ln(\lambda)}+1$, where λ is the largest absolute value of an eigenvalue of Γ different from 1. In our situation $|V|=k^n$ and by using spectrum shifting (achieved by adding loops) we obtain $d(n)\preceq\frac{n}{\delta(n)}$. In case of an expander family $d(n)\preceq Cn$, for some constant $C>0$, i.e. the diameter growth is linear. The converse is not true (the lamplighter and the Baumslag-Solitar examples have linear diameter growth). Examples of automaton groups producing either expanders or intermediate decay of the spectral gap are not known yet. Examples with exponential decay are provided by the first group in [10] and $H^{(3)}$. More generally, for contracting groups the spectral gap decays exponentially since the diameters grow exponentially. An example with polynomial decay is provided by the lamplighter group [13].

The rate of convergence of δ_n to 0 is of special interest and is related to the behavior of the KNS spectral measure of Γ and growth of Følner sets. The KNS measure ν is limit of counting measures ν_n on Γ_n [2] ($\nu_n(B)$ is the ratio $m(B)/k^n$, where $m(B)$ counts the eigenvalues of Γ_n in B).

We present here a full description of the spectrum of each 3-regular Schreier graph Γ_n modeling the Hanoi Towers Problem on 3 pegs as well as the spectrum of the 3-regular limit graph $\Gamma_{000\ldots}$ and the associated KNS spectral measure.

Theorem 2.3. *Let G be the Hanoi Towers group on three pegs $H^{(3)}$. The spectrum of Γ_n, as a set, has $3\cdot 2^{n-1}-1$ elements and is equal (after re-scaling by 3) to*

$$\{3\}\cup\bigcup_{i=0}^{n-1}f^{-i}(0)\cup\bigcup_{j=0}^{n-2}f^{-j}(-2),\tag{2}$$

where f is the polynomial $f(x)=x^2-x-3$. The multiplicity of the 2^i eigenvalues in $f^{-i}(0)$, $i=0,\ldots,n-1$, is a_{n-i} and the multiplicity of the 2^j eigenvalues in $f^{-j}(-2)$, $j=0,\ldots,n-2$, is b_{n-j}, where $a_i=\frac{3^{i-1}+3}{2}$ and $b_j=\frac{3^{j-1}-1}{2}$, for $i,j\geqslant 1$.

The spectrum of $\Gamma_{000\ldots}$, as a set, is equal (after re-scaling by 3) to

$$\overline{\{3\}\cup\bigcup_{i=0}^{\infty}f^{-i}\{0,-2\}}=\overline{\bigcup_{i=0}^{\infty}f^{-i}\{0\}}.\tag{3}$$

It consists of a set of isolated points $I=\bigcup_{i=0}^{\infty}f^{-i}\{0\}$ and its set of accumulation points J, which is the Julia set of the polynomial f and is a Cantor set. The KNS spectral measure is discrete, concentrated on $\bigcup_{i=0}^{\infty}f^{-i}\{0,-2\}$ and the measure of the eigenvalues in $f^{-i}\{0,-2\}$ is $\frac{1}{6\cdot 3^i}$, $i=0,1,\ldots$.

The method used to calculate the above spectra is a continuation of the direction set in [2]. At the core of this method lies a renormalization principle. The action of $H^{(3)}$ on level n of $\mathcal{T}$ induces 3^n-dimensional permutation matrix representation ρ_n. The spectra of Γ_n, i.e. of the adjacency matrices $T_n=\rho_n(a)+\rho_n(b)+\rho_n(c)$, are then calculated by considering two-dimensional pencils $\Delta_n(x,y)=T_n-xI_n+(y-1)K_n$ of matrices (where I_n is the identity matrix and K_n is additional matrix aiding the computation). The zeros of $\Delta_n(x,y)=0$ are related for various n by using iterations of a two-dimensional rational map $F:\mathbb{R}^2\to\mathbb{R}^2$. The spectrum of $\Gamma_{000\ldots}$ is the intersection of an invariant set of F (consisting of a family of hyperbolae) with the line $y=1$. The rational map F is semi-conjugate to the one-dimensional polynomial map $f(x)=x^2-x-3$ above, which makes possible the complete calculation of the spectra.

A similar result is obtained (by using a different method that also has the renormalization spirit in its root) in [17] for the spectrum of the infinite 4-regular (with single exception) graph approximating the Sierpiński gasket (the spectrum of this graph, as a set, was known since the 1980's from the work of Béllissard). The spectrum is obtained by iterations of a quadratic polynomial that is not conjugate to f. On the other hand, iterations of a polynomial conjugate to f appear in [9] in the functional equation for the Green's function of the graph obtained by gluing two copies of the graph from [17] along a vertex.

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