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Functional Analysis

The Banach algebra generated by a C_0-semigroup

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Abstract

Let $\mathbf{T} = \{T(t)\}_{t\geqslant 0}$ be a bounded C_0-semigroup on a Banach space with generator A. We define $A_{\mathbf{T}}$ as the closure with respect to the operator-norm topology of the set $\{\hat{f}(\mathbf{T})\colon f \in L^1(\mathbb{R}_+)\}$, where $\hat{f}(\mathbf{T}) = \int_0^\infty f(t)T(t)\,\mathrm{d}t$ is the Laplace transform of $f \in L^1(\mathbb{R}_+)$ with respect to the semigroup $\mathbf{T}$. Then $A_{\mathbf{T}}$ is a commutative Banach algebra. It is shown that if the unitary spectrum $\sigma(A)\cap \mathrm{i}\mathbb{R}$ of A is at most countable, then the Gelfand transform of $S \in A_{\mathbf{T}}$ vanishes on $\sigma(A)\cap \mathrm{i}\mathbb{R}$ if and only if, $\lim_{t\to\infty} \|T(t)S\| = 0$. Some applications to the semisimplicity problem are given. ***To cite this article: H. Mustafayev, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

L'algebre de Banach engendrée par un C_0-semigroupe. Soit $\mathbf{T} = \{T(t)\}_{t\geqslant 0}$ un C_0-semigroupe borné dans un espace de Banach par générateur A. Nous définissons $A_{\mathbf{T}}$ comme la clotûre par rapport à la topologie de la norme opérateur de l'ensemble $\{\hat{f}(\mathbf{T})\colon f \in L^1(\mathbb{R}_+)\}$, où $\hat{f}(\mathbf{T}) = \int_0^\infty f(t)T(t)\,\mathrm{d}t$ est la transformée de Laplace de $f \in L^1(\mathbb{R}_+)$ par rapport au semigroupe $\mathbf{T}$. Alors $A_{\mathbf{T}}$ est une algèbre de Banach commutative. Dans cet article il est montré que, si la spectre unitaire $\sigma(A)\cap \mathrm{i}\mathbb{R}$ de A est au plus dénombrable, alors la transformée de Gelfand de $S \in A_{\mathbf{T}}$ s'annule sur $\sigma(A)\cap \mathrm{i}\mathbb{R}$ si et seulement si $\lim_{t\to\infty} \|T(t)S\| = 0$. Nous donnons aussi quelques applications de la semisimplicité du problème. ***Pour citer cet article : H. Mustafayev, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction and preliminaries

Let X be complex Banach space and $B(X)$ the algebra of all bounded, linear operators on X. A family $\mathbf{T} = \{T(t)\}_{t\geqslant 0}$ in $B(X)$ is called a C_0-*semigroup*, if the following properties are satisfied: (1) $T(0) = I$, the identity operator on X; (2) $T(t+s) = T(t)T(s)$, for every $t, s \geqslant 0$; (3) $\lim_{t\to 0^+} \|T(t)x - x\| = 0$, for all $x \in X$.

The *generator* of the C_0-semigroup $\mathbf{T} = \{T(t)\}_{t\geqslant 0}$ is the linear operator A with domain $D(A)$ defined by

$$Ax = \lim_{t\to 0^+} \frac{1}{t}\big(T(t)x - x\big), \quad x \in D(A).$$

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The generator is always a closed, densely defined operator. The C_0-*groups* are defined analogously to C_0-semigroups. A C_0-semigroup $\mathbf{T} = \{T(t)\}_{t\geqslant 0}$ will be said to be bounded if $\sup_{t\geqslant 0} \|T(t)\| < \infty$.

Let $\mathbf{T} = \{T(t)\}_{t\geqslant 0}$ be a bounded C_0-semigroup with generator A. Then the spectrum $\sigma(A)$ of A belongs to the closed left half-plane. $\sigma(A) \cap \mathrm{i}\mathbb{R}$ is called the *unitary spectrum* of A.

The Fourier transform $\hat{f}(z)$ of $f \in L^1(\mathbb{R}_+)$, where $\hat{f}(z) = \int_0^\infty \exp(-\mathrm{i}tz) f(t)\,\mathrm{d}t$ is a function analytic in the open half-plane $\{z \in \mathbb{C},\ \operatorname{Im} z < 0\}$ and is a bounded continuous function in the closed half-plane $\{z \in \mathbb{C},\ \operatorname{Im} z \leqslant 0\}$. For a function $f \in L^1(\mathbb{R}_+)$, we put

$$\hat{f}(\mathbf{T}) = \int_0^\infty f(t)T(t)\,\mathrm{d}t.$$

The map $f \to \hat{f}(\mathbf{T})$ is a continuous homomorphism from $L^1(\mathbb{R}_+)$ into $B(X)$. We define $A_\mathbf{T}$ as the closure with respect to the operator-norm topology of the set $\{\hat{f}(\mathbf{T}):\ f \in L^1(\mathbb{R}_+)\}$. Then $A_\mathbf{T}$ is a commutative Banach algebra. The maximal ideal space of $A_\mathbf{T}$ will be denoted by $M_\mathbf{T}$. If $S \in A_\mathbf{T}$, its Gelfand transform will be denoted as $\widehat{S}$. It can be easily verified that the map $z \to \phi_z$, homeomorphically identifies $\sigma(A)$ with a closed subset of $M_\mathbf{T}$, where $\phi_z : A_\mathbf{T} \to \mathbb{C}$ is defined by $\phi_z(\hat{f}(\mathbf{T})) = \hat{f}(\mathrm{i}z)$. Therefore, instead of $\widehat{S}(\phi_z)(= \phi_z(S))$, $z \in \sigma(A)$, we can (and will) write $\widehat{S}(z)$.

Note that if $(h_n)_{n\in\mathbb{N}}$ is a bounded approximate identity (b.a.i.) for $L^1(\mathbb{R}_+)$, then $(\hat{h}_n(\mathbf{T}))_{n\in\mathbb{N}}$ is a b.a.i. for $A_\mathbf{T}$. Let $B_\mathbf{T}(X)$ be the closed subspace of $B(X)$ consisting of all $Q \in B(X)$ such that $\lim_{t\to 0^+} \|T(t)Q - Q\| = 0$. It is easily checked that $B_\mathbf{T}(X)$ contains $A_\mathbf{T}$. For $T \in B(X)$, we denote by $\widetilde{T}$ the left multiplication operator on $B(X)$. Then $\widetilde{\mathbf{T}} = \{\widetilde{T}(t)\}_{t\geqslant 0}$ is a C_0-semigroup on $B_\mathbf{T}(X)$. Let $\tilde{A}$ denote its generator. As is known [5, Lemma 5.2.2], $\sigma(\tilde{A}) \subset \sigma(A)$.

2. The main result

Recall [5, p. 163] that the function $f \in L^1(\mathbb{R}_+)$ is said to be of *spectral synthesis* with respect to a closed subset K of $\mathbb{R}$ if it can be approximated (in L^1-norm) by functions $g_n \in L^1(\mathbb{R})$ such that $\hat{g}_n = 0$ on a neighborhood of K. Semigroup version of the Katznelson–Tzafriri Theorem (see, [3,6] and [5, Theorem 5.2.3]) asserts that if $f \in L^1(\mathbb{R}_+)$ which is of spectral synthesis with respect to $\mathrm{i}\sigma(A) \cap \mathbb{R}$ (in particular, if $\sigma(A) \cap \mathrm{i}\mathbb{R}$ is at most countable and $\hat{f}(z)$ vanishes on $\sigma(A) \cap \mathrm{i}\mathbb{R}$), then $\lim_{t\to\infty} \|T(t)\hat{f}(\mathbf{T})\| = 0$.

The main result of this Note is the following theorem:

Theorem 2.1. *Let $\mathbf{T} = \{T(t)\}_{t\geqslant 0}$ be a C_0-semigroup of contractions on a Banach space X with generator A. If the unitary spectrum $\sigma(A) \cap \mathrm{i}\mathbb{R}$ of A is at most countable, then the Gelfand transform of $S \in A_\mathbf{T}$ vanishes on $\sigma(A) \cap \mathrm{i}\mathbb{R}$ if and only if* $\lim_{t\to\infty} \|T(t)S\| = 0$.

For the proof we need some preliminary results.

Recall that the w^*-*spectrum* $\sigma_*(\varphi)$ of $\varphi \in L^\infty(\mathbb{R})$ is defined as the hull of the closed ideal $I_\varphi = \{f \in L^1(\mathbb{R}):\ \varphi * f = 0\}$. Let $AP(\mathbb{R})$ be the space of all almost periodic functions on $\mathbb{R}$ and let Φ be the invariant mean on $AP(\mathbb{R})$. For $\lambda \in \mathbb{R}$, let $C_\lambda(\varphi)$ denote the Fourier–Bohr coefficient of a function $\varphi \in AP(\mathbb{R})$; $C_\lambda(\varphi) = \Phi[\exp(-\mathrm{i}\lambda t)\varphi(t)]$. The *Bohr spectrum* $\sigma_B(\varphi)$ of $\varphi \in AP(\mathbb{R})$ is defined as the set of all $\lambda \in \mathbb{R}$ such that $C_\lambda(\varphi) \neq 0$. As is well known if $\varphi \in AP(\mathbb{R})$ then $\sigma_B(\varphi) \subseteq \sigma_*(\varphi)$ and moreover, $\sigma_*(\varphi) = \overline{\sigma_B(\varphi)}$. We also note that if $\varphi \in AP(\mathbb{R})$ and $f \in L^1(\mathbb{R})$, then $\varphi * f \in AP(\mathbb{R})$ and $C_\lambda(\varphi * f) = \hat{f}(\lambda)C_\lambda(\varphi)$. It follows that $\sigma_B(\varphi * f) = \{\lambda \in \sigma_B(\varphi):\ \hat{f}(\lambda) \neq 0\}$. Well known Loomis Theorem [4] states that, bounded uniformly continuous function with countable w^*-spectrum is almost periodic.

The following result is contained in [5] (Theorem 5.1.2 and Corollary 5.1.3):

Lemma 2.2. *Let $\mathbf{T} = \{T(t)\}_{t\geqslant 0}$ be a C_0-semigroup of contractions on a Banach space X, with generator A such that $\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \mathrm{i}\mathbb{R}$. If $\inf_{t\geqslant 0} \|T(t)x\| > 0$ for some $x \in X\backslash\{0\}$, then there exist a Banach space $Y \neq \{0\}$, a bounded linear operator $J : X \to Y$, with dense range and a C_0-group of isometries $\mathbf{U} = \{U(t)\}_{t\in\mathbb{R}}$ on Y with generator B such that*: (i) $\|Jx\| = \lim_{t\to\infty} \|T(t)x\|$ *for all* $x \in X$; (ii) $U(t)J = JT(t)$ *for all* $t \geqslant 0$; (iii) $\sigma(B) \subset \sigma(A) \cap \mathrm{i}\mathbb{R}$.

Another result we need for the proof of the theorem is the following lemma:

Lemma 2.3. *Let $\mathbf{U} = \{U(t)\}_{t\in\mathbb{R}}$ be a C_0-group of isometries on a Banach space Y with generator B such that $\sigma(B)$ is at most countable. Then for every nonzero $\phi \in Y^*$, there exist a Hilbert space $H_\phi \neq \{0\}$, a bounded linear operator $J_\phi : Y \to H_\phi$ with dense range and a C_0-group of unitary operators $\mathbf{U}_\phi = \{U_\phi(t)\}_{t\in\mathbb{R}}$ on H_ϕ with generator B_ϕ such that*: (i) $\bigcap_{\phi\in Y^*} \ker J_\phi = \{0\}$; (ii) $U_\phi(t)J_\phi = J_\phi U(t)$ *for all* $t \in \mathbb{R}$; (iii) $\sigma(B_\phi) \subset \sigma(B)$.

Proof. Let $\phi \in Y^*\backslash\{0\}$ be given. For any $y \in Y$, we define the function $\bar{y}_\phi(t)$ on $\mathbb{R}$ by $\bar{y}_\phi(t) = \phi((U(-t))y)$. Then $\bar{y}_\phi(t)$ is a bounded continuous function and $\|\bar{y}_\phi\|_\infty \leqslant \|\phi\|\|y\|$. We claim that $\bar{y}_\phi(t) \in AP(\mathbb{R})$. Since the set $\{\hat{f}(\mathbf{U})y\colon\ y \in Y,\ f \in L^1(\mathbb{R})\}$ is dense in Y, we may assume that y is of the form $\hat{f}(\mathbf{U})z$, for some $f \in L^1(\mathbb{R})$ and $z \in Y$. Also, since

$$\bar{y}_\phi(t) = \phi\big(U(-t)\hat{f}(\mathbf{U})z\big) = \int_{\mathbb{R}} \phi\big(U(s-t)z\big) f(s)\,\mathrm{d}s = \int_{\mathbb{R}} \bar{z}_\phi(t-s) f(s)\,\mathrm{d}s = (\bar{z}_\phi * f)(t),$$

$\bar{y}_\phi(t)$ is a uniformly continuous function. Now, let us show that $\sigma_*(\bar{y}_\phi) \subset \sigma(\mathrm{i}B)$. Assume that there is a $\lambda_0 \in \sigma_*(\bar{y}_\phi)$, but $\lambda_0 \notin \sigma(\mathrm{i}B)$. Then there exists a $f \in L^1(\mathbb{R})$ such that $\hat{f}(\lambda_0) \neq 0$ and $\hat{f}(\lambda) = 0$ in a neighborhood of $\sigma(\mathrm{i}B)$. Hence, f belongs to the smallest closed ideal in $L^1(\mathbb{R})$ whose hull is $\sigma(\mathrm{i}B)$. Using the fact (see, [1, p. 223] and [2]) that

$$\operatorname{hull}\big\{f \in L^1(\mathbb{R})\colon\ \hat{f}(\mathbf{U}) = 0\big\} = \sigma(\mathrm{i}B),$$

we have $\hat{f}(\mathbf{U}) = 0$, so that $\bar{y}_\phi * f = 0$. Since $\lambda_0 \in \sigma_*(\bar{y}_\phi)$, it follows that $\hat{f}(\lambda_0) = 0$. This contradicts $\hat{f}(\lambda_0) \neq 0$. Hence, $\sigma_*(\bar{y}_\phi)$ is at most countable. By the Loomis Theorem [4], $\bar{y}_\phi(t) \in AP(\mathbb{R})$. This proves the claim.

Let H_ϕ^0 denote the linear set $\{\bar{y}_\phi(t)\colon\ y \in Y\}$ with the inner product defined by

$$\langle \bar{y}_\phi, \bar{z}_\phi \rangle = \Phi\big[\bar{y}_\phi(t)\overline{\bar{z}_\phi(t)}\big] \quad (y, z \in Y),$$

and let H_ϕ be the completion of H_ϕ^0 with respect to this inner product. Since $\|\bar{y}_\phi\| \leqslant \|\bar{y}_\phi\|_\infty \leqslant \|\phi\|\|y\|$, the map $J_\phi : Y \to H_\phi$, defined by $J_\phi y = \bar{y}_\phi$ is a bounded linear operator with dense range. It is easy to see that $\bigcap_{\phi\in Y^*} \ker J_\phi = \{0\}$. Now let $\mathbf{U}_\phi = \{U_\phi(t)\}_{t\in\mathbb{R}}$ be the translation group on H_ϕ; $U_\phi(t)\bar{y}_\phi(s) = \bar{y}_\phi(t-s)$. Then $\mathbf{U}_\phi = \{U_\phi(t)\}_{t\in\mathbb{R}}$ is a C_0-group of unitary operators on H_ϕ and $U_\phi(t)J_\phi = J_\phi U(t)$ for all $t \in \mathbb{R}$. We have proved (i) and (ii).

Next we prove (iii). Let B_ϕ be the generator of $\mathbf{U}_\phi$. Let $f \in L^1(\mathbb{R})$ and $y \in Y$ be given. Since $\hat{f}(\mathbf{U}_\phi)J_\phi y = J_\phi \hat{f}(\mathbf{U})y = \bar{y}_\phi * f$ and $C_\lambda(\bar{y}_\phi * f) = \hat{f}(\lambda)C_\lambda(\bar{y}_\phi)$, it follows from the Parseval's identity that

$$\begin{aligned}\big\|\hat{f}(\mathbf{U}_\phi)J_\phi y\big\| &= \Bigg(\sum_{\lambda\in\sigma_B(\bar{y}_\phi * f)} \big|\hat{f}(\lambda)\big|^2 \big|C_\lambda(\bar{y}_\phi)\big|^2\Bigg)^{1/2} \leqslant \sup_{\lambda\in\sigma_B(\bar{y}_\phi)} \big|\hat{f}(\lambda)\big|\|\bar{y}_\phi\| \leqslant \sup_{\lambda\in\sigma(\mathrm{i}B)} \big|\hat{f}(\lambda)\big|\|\bar{y}_\phi\| \\ &\leqslant \big\|\hat{f}(\mathbf{U})\big\|\|J_\phi y\|.\end{aligned}$$

Also since J_ϕ has dense range, we obtain $\|\hat{f}(\mathbf{U}_\phi)\| \leqslant \|\hat{f}(\mathbf{U})\|$. It follows that the map $\hat{f}(\mathbf{U}) \to \hat{f}(\mathbf{U}_\phi)$ can be extended continuously to a norm decreasing homomorphism $h : A_{\mathbf{U}} \to A_{\mathbf{U}_\phi}$. It can be seen that $h^* M_{\mathbf{U}_\phi} \subset M_{\mathbf{U}}$. Now, using the fact (see, [1, p. 223] and [2]) that $M_{\mathbf{U}_\phi} = \sigma(\mathrm{i}B_\phi)$ and $M_{\mathbf{U}} = \sigma(\mathrm{i}B)$, we obtain $\sigma(B_\phi) \subset \sigma(B)$. □

Proof of Theorem 2.1. Assume that for some $S \in A_{\mathbf{T}}$, $\lim_{t\to\infty} \|T(t)S\| \to 0$. Let $t \in \mathbb{R}$, $\mathrm{i}y \in \sigma(A)$ $(y \in \mathbb{R})$, $f \in L^1(\mathbb{R}_+)$ and let

$$f_t(s) = \begin{cases} f(s-t), & s \geqslant t, \\ 0, & 0 \leqslant s < t. \end{cases}$$

Then we can write

$$\phi_{\mathrm{i}y}\big(T(t)\hat{f}(\mathbf{T})\big) = \phi_{\mathrm{i}y}\big(\hat{f}_t(\mathbf{T})\big) = \hat{f}_t(-y) = \exp(\mathrm{i}yt)\hat{f}(-y) = \exp(\mathrm{i}yt)\phi_{\mathrm{i}y}\big(\hat{f}(\mathbf{T})\big).$$

Since $\{\hat{f}(\mathbf{T})\colon\ f \in L^1(\mathbb{R}_+)\}$ is dense in $A_{\mathbf{T}}$, we have $\phi_{\mathrm{i}y}(T(t)S) = \exp(\mathrm{i}yt)\widehat{S}(\mathrm{i}y)$. It follows that

$$\big|\widehat{S}(\mathrm{i}y)\big| = \big|\phi_{\mathrm{i}y}\big(T(t)S\big)\big| \leqslant \big\|T(t)S\big\| \to 0, \quad t \to \infty.$$

Now let $S \in A_{\mathbf{T}}$ be such that $\widehat{S}(z) \equiv 0$ on $\sigma(A) \cap \mathrm{i}\mathbb{R}$. First we prove that $\lim_{t\to\infty} \|T(t)Sx\| = 0$ for all $x \in X$. If $\lim_{t\to\infty} \|T(t)x\| = 0$ for all $x \in X$, then there is nothing to prove. Hence, we may assume that $\inf_{t\geqslant 0} \|T(t)x\| > 0$ for some $x \in X\backslash\{0\}$. By Lemma 2.2 there exist a Banach space $Y \neq \{0\}$, a bounded linear operator $J : X \to Y$ with dense range and a C_0-group of isometries $\mathbf{U} = \{U(t)\}_{t\in\mathbb{R}}$ on Y with generator B such that: (i) $\|Jx\| = \lim_{t\to\infty} \|T(t)x\|$ for all $x \in X$; (ii) $U(t)J = JT(t)$ for all $t \geqslant 0$; (iii) $\sigma(B) \subset \sigma(A) \cap i\mathbb{R}$.

Let $\phi \in Y^*\backslash\{0\}$ be given. By Lemma 2.3 there exist a Hilbert space $H_\phi \neq \{0\}$, a bounded linear operator $J_\phi : Y \to H_\phi$, with dense range and a C_0-group of unitary operators $\mathbf{U}_\phi = \{U_\phi(t)\}_{t\in\mathbb{R}}$ on H_ϕ with generator B_ϕ such that: (iv) $\bigcap_{\phi\in Y^*} \ker J_\phi = \{0\}$; (v) $U_\phi(t)J_\phi = J_\phi U(t)$ for all $t \in \mathbb{R}$; (vi) $\sigma(B_\phi) \subset \sigma(B)$.

Since $S \in A_{\mathbf{T}}$, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1(\mathbb{R}_+)$ such that $\|\hat{f}_n(\mathbf{T}) - S\| \to 0$. It follows that $\phi_z(\hat{f}_n(\mathbf{T})) = \hat{f}_n(\mathrm{i}z) \to 0$-uniformly for $z \in \sigma(A) \cap \mathrm{i}\mathbb{R}$. From (iii) and (vi) we have $\sigma(B_\phi) \subset \sigma(A) \cap \mathrm{i}\mathbb{R}$, so that $\hat{f}_n(\mathrm{i}z) \to 0$-uniformly for $z \in \sigma(B_\phi)$. Since for every $f \in L^1(\mathbb{R}_+)$,

$$\|\hat{f}(\mathbf{U}_\phi)\| = \sup_{z\in\sigma(B_\phi)} |\hat{f}(\mathrm{i}z)|,$$

it follows that $\|\hat{f}_n(\mathbf{U}_\phi)\| \to 0$. Further, from (ii) and (v) we can write $U_\phi(t)J_\phi J = J_\phi J T(t)$ $(t \geqslant 0)$, which implies that $\hat{f}_n(\mathbf{U}_\phi)J_\phi J = J_\phi J \hat{f}_n(\mathbf{T})$. By taking the limit as $n \to \infty$, we obtain $J_\phi J S = 0$ for all $\phi \in Y^*$. It follows from (iv) that $JS = 0$. By (i) this means that $\lim_{t\to\infty} \|T(t)Sx\| = 0$ for every $x \in X$.

Next, let $\widetilde{\mathbf{T}} = \{\widetilde{T}(t)\}_{t\geqslant 0}$ be a C_0-semigroup on $B_{\mathbf{T}}(X)$ with generator $\tilde{A}$. Note that $\widetilde{S} \in A_{\tilde{\mathbf{T}}}$. Since $\sigma(\tilde{A}) \subset \sigma(A)$, the set $\sigma(\tilde{A}) \cap \mathrm{i}\mathbb{R}$ is at most countable. On the other hand, $\widehat{\widetilde{S}}(z) = \widehat{S}(z) = 0$ for all $z \in \sigma(\tilde{A}) \cap \mathrm{i}\mathbb{R}$. Now, it follows from what is proved above that $\lim_{t\to\infty} \|\widetilde{T}(t)\widetilde{S}Q\| = 0$ for all $Q \in B_{\mathbf{T}}(X)$. In particular, we have $\lim_{t\to\infty} \|T(t)SQ\| = 0$ for all $Q \in A_{\mathbf{T}}$. Let $(Q_n)_{n\in\mathbb{N}}$ be a b.a.i. for $A_{\mathbf{T}}$. Then for any $\varepsilon > 0$ and for some $n \in \mathbb{N}$, we have $\|SQ_n - S\| < \varepsilon$. This implies $\|T(t)S\| < \|T(t)SQ_n\| + \varepsilon$ for all $t \geqslant 0$. As $t \to \infty$, we obtain that $\lim_{t\to\infty} \|T(t)S\| < \varepsilon$. □

3. Semisimplicity

Let $\boldsymbol{A}$ be a complex commutative Banach algebra. If the Gelfand transform on $\boldsymbol{A}$ is injective, then $\boldsymbol{A}$ is said to be *semisimple*. If $\boldsymbol{A}$ is a closed commutative subalgebra of $B(X)$, then $\boldsymbol{A}$ is semisimple if and only if it does not contain a nonzero quasi-nilpotent operator.

The following example shows that there exists a uniformly bounded C_0-semigroup with one-point unitary spectrum that generates a non-semisimple algebra.

Example 3.1. Let V be the Volterra operator on the Hilbert space $L^2[0, 1]$, defined by $(Vf)(t) = \int_0^t f(s)\,\mathrm{d}s$ and let $\mathbf{T} = \{\mathrm{e}^{-tV}\}_{t\geqslant 0}$. Notice that the exponential formula, yields $\|\mathrm{e}^{-tV}\| = 1$ for all $t \geqslant 0$. On the other hand, V is a nonzero quasi-nilpotent operator and $V \in A_{\mathbf{T}}$. Hence, $A_{\mathbf{T}}$ is not semisimple.

As an immediate corollary of the Theorem 2.1 we have the next result.

Corollary 3.2. *Let* $\mathbf{T} = \{T(t)\}_{t\geqslant 0}$ *be a bounded* C_0*-semigroup on a Banach space* X *with generator* A *such that the unitary spectrum* $\sigma(A) \cap \mathrm{i}\mathbb{R}$ *of* A *is at most countable and* $\inf_{t\geqslant 0} \|T(t)x\| > 0$ *for all* $x \in X\backslash\{0\}$*. If the Gelfand transform of* $S \in A_{\mathbf{T}}$ *vanishes on* $\sigma(A) \cap \mathrm{i}\mathbb{R}$*, then* $S = 0$*. In particular,* $A_{\mathbf{T}}$ *is semisimple.*

References

[1] W. Arveson, The harmonic analysis of automorphism groups of operator algebras, Proc. Sympos. Pure Math. 38 (1982) 199–269.
[2] D.E. Evans, On the spectrum of a one-parameter strongly continuous representation, Math. Scand. 39 (1976) 80–82.
[3] Y. Katznelson, L. Tzafriri, On power bounded operators, J. Funct. Anal. 68 (1986) 313–328.
[4] L.H. Loomis, The spectral characterization of a class of almost periodic functions, Ann. Math. 72 (1960) 362–368.
[5] J.M.A.M. van Neerven, The Asymptotic Behavior of Semigroups of Linear Operators, Birkhäuser, 1996.
[6] V.Q. Phong, Theorems of Katznelson–Tzafriri type for semigroups of operators, J. Funct. Anal. 103 (1992) 74–84.