Available online at www.sciencedirect.com
SCIENCE DIRECT®

C. R. Acad. Sci. Paris, Ser. I 342 (2006) 579–584

COMPTES RENDUS MATHEMATIQUE

http://france.elsevier.com/direct/CRASS1/

Optimal Control

A weighted identity for partial differential operators of second order and its applications

Xiaoyu Fu

School of Mathematics, Sichuan University, Chengdu 610064, China

Received 8 August 2005; accepted 21 February 2006

Available online 20 March 2006

Presented by Philippe G. Ciarlet

Abstract

In this Note, a weighted identity for partial differential operators of second order is established. As its applications, one may deduce all the known controllability/observability results for the parabolic, hyperbolic, Schrödinger and plate equations that are derived via Carleman estimate. Meanwhile, a new controllability/observability result is presented for the parabolic equations with a complex principal part. ***To cite this article: X. Fu, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Une identité avec poids pour des opérateurs aux dérivées partielles du second ordre et ses applications. Dans cette Note, nous établissons une identité avec poids pour des opérateurs aux dérivées partielles du second ordre. De cette égalité, découlent tous les resultats connus de contrôlabilité/observabilité pour les équations paraboliques, les équations hyperboliques, l'équation de Schrödinger et celle des plaques, tous obtenus à partir des inégalités de Carleman. Par ailleurs, un nouveau résultat de contrôlabilité/observabilité est obtenu pour les équations de type paraboliques avec des coefficients à valeur complexe. ***Pour citer cet article : X. Fu, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Nous utiliserons la notation $\frac{\partial}{\partial x_j} f = f_j$, $\frac{\partial}{\partial x_j}\frac{\partial}{\partial t} f = f_{tj}$, $\sum_j = \sum_{j=1}^n$, $\sum_{j,k} = \sum_{j,k=1}^n$, etc., où x_j est la $j^{\text{ième}}$ coordonnée de $x \in \mathbb{R}^n$, $n \in \mathbb{N}$. Pour tout $c \in \mathbb{C}$, le nombre complexe conjugué de c sera noté $\bar{c}$.

Soit

$$\mathcal{P}w \stackrel{\triangle}{=} (a + \mathrm{i}b)w_t + \sum_{j,k} \big(a^{jk} w_j\big)_k, \quad \forall a, b \in \mathbb{R}, \quad \mathrm{i} = \sqrt{-1}. \tag{1}$$

Le principal résultat de cette Note est décrit comme suit :

E-mail address: rj_xy@163.com (X. Fu).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2006.02.023

Théorème 0.1. *Soit $a^{jk} \in C^{1,2}(\mathbb{R}^{1+n};\mathbb{R})$ tel que $a^{jk}=a^{kj}$, $j,k=1,2,\ldots,n$. Soient $w \in C^2(\mathbb{R}^{1+n};\mathbb{C})$, $\ell \in C^3(\mathbb{R}^{1+n};\mathbb{R})$, $\theta = \mathrm{e}^{\ell}$ et $v=\theta w$. Alors,*

$$
\begin{aligned}
&\theta\big(\mathcal{P}w\overline{I_1}+\overline{\mathcal{P}w}I_1\big)+\Big\{\Big[\big(a^2+b^2\big)\ell_t-a\sum_{j,k}a^{jk}\ell_j\ell_k\Big]|v|^2+a\sum_{j,k}a^{jk}v_j\bar v_k+\mathrm{i}b\sum_{j,k}a^{jk}\ell_j(\bar v_k v-v_k\bar v)\Big\}_t\\
&\quad+\sum_{j,k,j',k'}\big\{-\mathrm{i}b\big[a^{jk}\ell_j(\bar v_t v-\bar v v_t)+a^{jk}\ell_t(v_j\bar v-\bar v_j v)\big]\\
&\quad-aa^{jk}(v_j\bar v_t+\bar v_j v_t)+\big(2a^{jk'}a^{j'k}-a^{jk}a^{j'k'}\big)\ell_j(v_{j'}\bar v_{k'}+\bar v_{j'}v_{k'})\\
&\quad+\big(a^{j'k'}\ell_{j'}\big)_{k'}a^{jk}(v_j\bar v+\bar v_j v)+a^{jk}\big(2a^{j'k'}\ell_j\ell_{j'}\ell_{k'}-\big(a^{j'k'}\ell_{j'}\big)_{k'j}-2a\ell_j\ell_t\big)|v|^2\big\}_k\\
&=2|I_1|^2+\sum_{j,k,j',k'}\Big[2\big(a^{j'k}\ell_{j'}\big)_{k'}a^{jk'}-a^{jk}_{k'}a^{j'k'}\ell_{j'}+\frac{a}{2}a^{jk}_t\Big](v_k\bar v_j+\bar v_k v_j)\\
&\quad+\mathrm{i}b\sum_{j,k}\big(a^{jk}_t\ell_j+2a^{jk}\ell_{jt}\big)(\bar v_k v-v_k\bar v)+B|v|^2,
\end{aligned}
\tag{2}
$$

où

$$
\begin{cases}
I_1\stackrel{\triangle}{=}\mathrm{i}bv_t-a\ell_t v+\sum_{j,k}\big(a^{jk}v_j\big)_k+\sum_{j,k}a^{jk}\ell_j\ell_k v,\\
B\stackrel{\triangle}{=}\big(a^2+b^2\big)\ell_{tt}-a\sum_{j,k}\big[\big(a^{jk}\ell_j\ell_k\big)_t+2a^{jk}\ell_j\ell_{tk}\big]\\
\qquad+2\sum_{j,k,j',k'}a^{jk}\ell_j\big(a^{j'k'}\ell_{j'}\ell_{k'}\big)_k-\sum_{j,k,j',k'}\big[a^{jk}\big(a^{j'k'}\ell_{j'}\big)_{k'k}\big]_j.
\end{cases}
\tag{3}
$$

Remarque 1. En choisissant $a=1$ et $b=0$ dans Théorème 0.1, on obtient une identité à poids pour l'opérateur parabolique. On peut ainsi retrouver tous les résultats de contrôlabilité/observabilité pour les équations paraboliques dans [2] et [5], en suivant [7].

Remarque 2. En choisissant $a^{jk}(t,x)\equiv a^{jk}(x)$ et $a=b=0$ dans Théorème 0.1, on obtient l'identité décrite dans [4] qui implique des résultats de contrôlabilité/observabilité pour les équations hyperboliques.

Remarque 3. En choisissant $(a^{jk})_{1\leqslant j,k\leqslant n}$ comme étant la matrice identité, $a=0$ et $b=1$ dans Théorème 0.1, on obtient l'identité décrite dans [6] qui implique des résultats d'observabilité pour les équations de type Schrödinger. Aussi, ceci amène les résultats de contrôlabilité/observabilité pour les équations des plaques dans [8] et les résultats pour le problème inverse des équations de Schrödinger dans [1].

Remarque 4. En choisissant $a<0$ et $(a^{jk})_{1\leqslant j,k\leqslant n}$ comme étant une matrice définie positive dans Théorème 0.1, il en découle une identité à poids pour les opérateurs paraboliques avec un coefficient complexe devant ∂_t. Ceci implique un nouveau résultat de contrôlabilité/observabilité pour les équations paraboliques de ce type.

Théorème 0.1 permet une approche globale pour les problèmes de contrôlabilité/observabilité des équations aux dérivées partielles du second ordre.

1. Introduction and main results

The study of controllability/observability problem for partial differential equations (PDEs for short) began in the 1960s, for which various techniques have been developed in the last decades [9,10]. In this respect, Carleman estimate is one of the most powerful tools, for both parabolic and hyperbolic-type equations [2,5,4,6,8]. However, the Carleman estimate that has been developed up to now to establish observability inequalities of PDEs depend heavily on the nature of the equations. In this Note, we present a point-wise weighted identity for partial differential operators of second order, by which we develop a unified approach, based on global Carleman estimate, to treat the controllability/observability problems for PDEs of second order.

In the following, we will use the notation $\frac{\partial}{\partial x_j} f = f_j$, $\frac{\partial}{\partial x_j}\frac{\partial}{\partial t} f = f_{tj}$, $\sum_j = \sum_{j=1}^n$, $\sum_{j,k} = \sum_{j,k=1}^n$, etc., where x_j is the jth coordinate of $x \in \mathbb{R}^n$, $n \in \mathbb{N}$. For any $c \in \mathbb{C}$, denote its complex conjugate by $\bar{c}$.

We introduce the following formal partial differential operator of second order:

$$\mathcal{P}w \stackrel{\triangle}{=} (a+\mathrm{i}b)w_t + \sum_{j,k}\bigl(a^{jk}w_j\bigr)_k, \quad \forall a,b\in\mathbb{R}, \quad \mathrm{i}=\sqrt{-1}. \tag{4}$$

The main result of this Note is as follows:

Theorem 1.1. *Let $a^{jk} \in C^{1,2}(\mathbb{R}^{1+n};\mathbb{R})$ satisfy $a^{jk}=a^{kj}$, $j,k=1,2,\dots,n$. Let $w\in C^2(\mathbb{R}^{1+n};\mathbb{C})$, $\ell\in C^3(\mathbb{R}^{1+n};\mathbb{R})$, $\theta = \mathrm{e}^{\ell}$ and $v=\theta w$. Then*

$$\begin{aligned}
&\theta\bigl(\mathcal{P}w\overline{I_1}+\overline{\mathcal{P}w}I_1\bigr) + \Biggl\{\Bigl[\bigl(a^2+b^2\bigr)\ell_t - a\sum_{j,k}a^{jk}\ell_j\ell_k\Bigr]|v|^2 + a\sum_{j,k}a^{jk}v_j\bar{v}_k + \mathrm{i}b\sum_{j,k}a^{jk}\ell_j(\bar{v}_k v - v_k\bar{v})\Biggr\}_t \\
&\quad + \sum_{j,k,j',k'}\bigl\{-\mathrm{i}b\bigl[a^{jk}\ell_j(\bar{v}_t v-\bar{v}v_t) + a^{jk}\ell_t(v_j\bar{v}-\bar{v}_j v)\bigr] \\
&\quad - aa^{jk}(v_j\bar{v}_t+\bar{v}_j v_t) + \bigl(2a^{jk'}a^{j'k}-a^{jk}a^{j'k'}\bigr)\ell_j(v_{j'}\bar{v}_{k'}+\bar{v}_{j'}v_{k'}) \\
&\quad + \bigl(a^{j'k'}\ell_{j'}\bigr)_{k'}a^{jk}(v_j\bar{v}+\bar{v}_j v) + a^{jk}\bigl(2a^{j'k'}\ell_j\ell_{j'}\ell_{k'} - \bigl(a^{j'k'}\ell_{j'}\bigr)_{k'j} - 2a\ell_j\ell_t\bigr)|v|^2\bigr\}_k \\
&= 2|I_1|^2 + \sum_{j,k,j',k'}\Bigl[2\bigl(a^{j'k}\ell_{j'}\bigr)_{k'}a^{jk'} - a^{jk}_{k'}a^{j'k'}\ell_{j'} + \frac{a}{2}a^{jk}_t\Bigr](v_k\bar{v}_j+\bar{v}_k v_j) \\
&\quad + \mathrm{i}b\sum_{j,k}\bigl(a^{jk}_t\ell_j + 2a^{jk}\ell_{jt}\bigr)(\bar{v}_k v - v_k\bar{v}) + B|v|^2,
\end{aligned} \tag{5}$$

where

$$\begin{cases}
I_1 \stackrel{\triangle}{=} \mathrm{i}bv_t - a\ell_t v + \sum_{j,k}\bigl(a^{jk}v_j\bigr)_k + \sum_{j,k}a^{jk}\ell_j\ell_k v, \\
B \stackrel{\triangle}{=} \bigl(a^2+b^2\bigr)\ell_{tt} - a\sum_{j,k}\bigl[\bigl(a^{jk}\ell_j\ell_k\bigr)_t + 2a^{jk}\ell_j\ell_{tk}\bigr] \\
\qquad + 2\sum_{j,k,j',k'}a^{jk}\ell_j\bigl(a^{j'k'}\ell_{j'}\ell_{k'}\bigr)_k - \sum_{j,k,j',k'}\bigl[a^{jk}\bigl(a^{j'k'}\ell_{j'}\bigr)_{k'k}\bigr]_j.
\end{cases} \tag{6}$$

Remark 1. By choosing $a=1$ and $b=0$ in Theorem 1.1, one obtains a weighted identity for the parabolic operator. By this and following [7], one may recover all the controllability/observability results for the parabolic equations in [2] and [5].

Remark 2. By choosing $a^{jk}(t,x)\equiv a^{jk}(x)$ and $a=b=0$ in Theorem 1.1 (and noting that only the symmetry condition is assumed for a^{jk} in the above), one obtains the identity derived in [4] for the controllability/observability results for the general hyperbolic equations.

Remark 3. By choosing $(a^{jk})_{1\leqslant j,k\leqslant n}$ to be the identity matrix, $a=0$ and $b=1$ in Theorem 1.1, one obtains the pointwise identity derived in [6] for the observability results for the nonconservative Schrödinger equations. Also, this yields the controllability/observability results in [8] for the plate equations and the results for inverse problem for the Schrödinger equations in [1].

A consequence of Theorem 1.1 is the following point-wise weighted inequality for the parabolic operator $(a+\mathrm{i}b)\partial_t + \sum_{j,k}\partial_k(a^{jk}\partial_j)$ (with a complex principal part).

Theorem 1.2. *Let $a<0$, $a^{jk}\in C^{1,2}(\mathbb{R}^{1+n};\mathbb{R})$ satisfy $a^{jk}=a^{kj}$, $j,k=1,2,\dots,n$. Let $w\in C^2(\mathbb{R}^{1+n};\mathbb{C})$, $\ell\in C^3(\mathbb{R}^{1+n};\mathbb{R})$, $\theta=\mathrm{e}^{\ell}$ and $v=\theta w$. Then*

$$
\begin{aligned}
&2\int_Q \theta^2\Big|(a+\mathrm{i}b)w_t+\sum_{j,k}\big(a^{jk}w_j\big)_k\Big|^2\,\mathrm{d}t\,\mathrm{d}x+\int_Q M_t\,\mathrm{d}t\,\mathrm{d}x+\sum_k\int_Q V_k\,\mathrm{d}t\,\mathrm{d}x\\
&\quad\geqslant\sum_{j,k}\int_Q c^{jk}(v_k\bar v_j+\bar v_k v_j)\,\mathrm{d}t\,\mathrm{d}x+\int_Q B|v|^2\,\mathrm{d}t\,\mathrm{d}x+\mathrm{i}b\int_Q\sum_{j,k}\big(a_t^{jk}\ell_j+2a^{jk}\ell_{jt}\big)(\bar v_k v-v_k\bar v)\,\mathrm{d}t\,\mathrm{d}x,
\end{aligned}\tag{7}
$$

where B is defined as in (6) *and*

$$
\begin{cases}
M\stackrel{\triangle}{=}\big(a^2+b^2\big)\ell_t|v|^2-a\sum\limits_{j,k}a^{jk}\ell_j\ell_k|v|^2+a\sum\limits_{j,k}a^{jk}v_j\bar v_k+\mathrm{i}b\sum\limits_{j,k}a^{jk}\ell_j(\bar v_k v-v_k\bar v)=\textit{real-valued},\\
V_k\stackrel{\triangle}{=}\sum\limits_{j,j',k'}\big\{-\mathrm{i}b\big[a^{jk}\ell_j(\bar v_t v-\bar v v_t)+a^{jk}\ell_t(v_j\bar v-\bar v_j v)\big]-aa^{jk}(v_j\bar v_t+\bar v_j v_t)\\
\qquad+\big(a^{j'k'}\ell_{j'}\big)_{k'}a^{jk}(v_j\bar v+\bar v_j v)+\big(2a^{jk'}a^{j'k}-a^{jk}a^{j'k'}\big)\ell_j(v_{j'}\bar v_{k'}+\bar v_{j'}v_{k'})\\
\qquad+a^{jk}\big(2a^{j'k'}\ell_j\ell_{j'}\ell_{k'}-\big(a^{j'k'}\ell_{j'}\big)_{k'j}-2a\ell_j\ell_t\big)|v|^2\big\}_k=\textit{real-valued},\\
c^{jk}\stackrel{\triangle}{=}\sum\limits_{j',k'}\Big[2\big(a^{j'k}\ell_{j'}\big)_{k'}a^{jk'}-a_{k'}^{jk}a^{j'k'}\ell_{j'}+\dfrac{a}{2}a_t^{jk}\Big]=\textit{real-valued}.
\end{cases}\tag{8}
$$

As an application of Theorem 1.2, we show the null controllability of the following parabolic equations with a complex principal part:

$$
\begin{cases}
(a+\mathrm{i}b)y_t+\sum\limits_{j,k}\big(a^{jk}y_j\big)_k=\chi_\omega(x)u(t,x) & \text{in } Q\stackrel{\triangle}{=}(0,T)\times\Omega,\\
y=0 & \text{on } \Sigma\stackrel{\triangle}{=}(0,T)\times\Gamma,\\
y(0)=y_0 & \text{in } \Omega,
\end{cases}\tag{9}
$$

where $T>0$, $\Omega\subset\mathbb{R}^n$ is a given bounded domain with a C^2 boundary Γ, and $\omega\neq\emptyset$ is a given subdomain of Ω. In system (9), $a<0$, $b\in\mathbb{R}$, $a^{jk}\in C^{1,2}(\overline{Q};\mathbb{R})$ is assumed to satisfy $a^{jk}=a^{kj}$ $(j,k=1,2,\dots,n)$ and for some constant $\beta>0$,

$$
\sum_{j,k}a^{jk}\xi_j\bar\xi_k\geqslant\beta|\xi|^2,\quad\forall(t,x,\xi)\equiv(t,x,\xi_1,\dots,\xi_n)\in\overline{Q}\times\mathbb{C}^n.\tag{10}
$$

The null controllability of system (9) with $b=0$ is well-known (e.g., [5]). However, to the author's best knowledge, the same problem but with $b\neq0$ is not solved in the previous literature.

By [5], one can find a function $\psi\in C^2(\overline{\Omega};\mathbb{R})$ such that $\psi>0$ in Ω, $\psi=0$ on Γ, and $\nabla\psi\neq0$ in $\Omega\setminus\overline{\omega}$. For any parameters $\lambda>1$ and $\mu>1$, choose

$$
\ell=\lambda\alpha,\quad\alpha(t,x)=t^{-1}(T-t)^{-1}\big(\mathrm{e}^{\mu\psi(x)}-\mathrm{e}^{2\mu|\psi|_{C(\overline{\Omega};\mathbb{R})}}\big),\quad\varphi(t,x)=t^{-1}(T-t)^{-1}\,\mathrm{e}^{\mu\psi(x)}.\tag{11}
$$

From Theorem 1.2, one obtains the following Carleman estimate for the operator $(a+\mathrm{i}b)\partial_t+\sum_{j,k}\partial_k(a^{jk}\partial_j)$:

Theorem 1.3. *Let $a<0$, $b\in\mathbb{R}$, and $a^{jk}\in C^{1,2}(\overline{Q};\mathbb{R})$ satisfy $a^{jk}=a^{kj}$ $(j,k=1,2,\dots,n)$ and* (10). *Then there is a $\mu_0>0$ such that for all $\mu\geqslant\mu_0$, one can find two constants $C=C(\mu)>0$ and $\lambda_1=\lambda_1(\mu)$ so that for all $w\in C([0,T];L^2(\Omega))\bigcap L^2(0,T;H_0^1(\Omega))$ and $f\in L^2(\overline{Q};\mathbb{C})$ with $(a+\mathrm{i}b)w_t+\sum_{j,k}(a^{jk}w_j)_k=f$, and for all $\lambda\geqslant\lambda_1$, it holds*

$$
\lambda^3\mu^4\int_Q\varphi^3\theta^2|w|^2\,\mathrm{d}t\,\mathrm{d}x+\lambda\mu^2\int_Q\varphi\theta^2|\nabla w|^2\,\mathrm{d}t\,\mathrm{d}x\leqslant C\Big[\int_Q\theta^2|f|^2\,\mathrm{d}t\,\mathrm{d}x+\lambda^3\mu^4\int_{(0,T)\times\omega}\varphi^3\theta^2|w|^2\,\mathrm{d}t\,\mathrm{d}x\Big].\tag{12}
$$

As an immediate consequence of Theorem 1.3, one obtains the null controllability of system (9):

Theorem 1.4. *Let $a<0$, $b\in\mathbb{R}$, and $a^{jk}\in C^{1,2}(\overline{Q};\mathbb{R})$ satisfy $a^{jk}=a^{kj}$ $(j,k=1,2,\dots,n)$ and* (10). *Then for any given $y_0\in L^2(\Omega)$, there is a control $u\in L^2((0,T)\times\omega)$ such that the weak solution*

$$y(\cdot)\in C\big([0,T];L^2(\Omega)\big)\cap L^2\big(0,T;H_0^1(\Omega)\big)$$

of system (9) *satisfies $y(T)=0$ in Ω.*

The rest of this Note is to outline the proof of Theorem 1.1. We refer to [3] for a detailed proof of the results in this Note and other related results.

2. Sketch of the proof of Theorem 1.1

Proof. Recalling (6) for I_1, a direct computation shows that $\theta\mathcal{P}w=I_1+I_2$, where

$$I_2\stackrel{\triangle}{=}av_t-\mathrm{i}b\ell_tv-2\sum_{j,k}a^{jk}\ell_jv_k-\sum_{j,k}\big(a^{jk}\ell_j\big)_kv.$$

Hence

$$\theta\big(\mathcal{P}w\overline{I_1}+\overline{\mathcal{P}w}I_1\big)=2|I_1|^2+\big(I_1\bar I_2+I_2\bar I_1\big).\tag{13}$$

By the definitions of I_1 and I_2, we have

$$\begin{aligned}
I_1\bar I_2+I_2\bar I_1={}&-b^2\big(\ell_t|v|^2\big)_t+b^2\ell_{tt}|v|^2-\mathrm{i}b\sum_{j,k}\big[a^{jk}\ell_j(\bar v_kv-v_k\bar v)\big]_t\\
&+\mathrm{i}b\sum_{j,k}\big[a^{jk}\ell_j(\bar v_tv-v_t\bar v)\big]_k+\mathrm{i}b\sum_{j,k}\big(a^{jk}\ell_j\big)_t(\bar v_kv-v_k\bar v)-a^2\big(\ell_t|v|^2\big)_t+a^2\ell_{tt}|v|^2\\
&+2a\sum_{j,k}\big(a^{jk}\ell_j\ell_t|v|^2\big)_k-2a\sum_{j,k}a^{jk}\ell_j\ell_{tk}|v|^2+a\sum_{j,k}\big[a^{jk}(v_j\bar v_t+\bar v_jv_t)\big]_k\\
&-a\sum_{j,k}\big(a^{jk}v_j\bar v_k\big)_t+\frac{a}{2}\sum_{j,k}a_t^{jk}(v_j\bar v_k+v_k\bar v_j)+\mathrm{i}b\sum_{j,k}\big[a^{jk}\ell_t(v_j\bar v-\bar v_jv)\big]_k\\
&+\mathrm{i}b\sum_{j,k}a^{jk}\ell_{jt}(\bar v_kv-v_k\bar v)-2\sum_{j,k,j',k'}\big[a^{jk}a^{j'k'}\ell_j(v_{j'}\bar v_k+\bar v_{j'}v_k)\big]_{k'}\\
&+\sum_{j,k,j',k'}\big[a^{jk}a^{j'k'}\ell_j(v_{j'}\bar v_{k'}+\bar v_{j'}v_{k'})\big]_k-\sum_{j,k,j',k'}\big(a^{jk}a^{j'k'}\ell_j\big)_k(v_{j'}\bar v_{k'}+\bar v_{j'}v_{k'})\\
&+2\sum_{j,k,j',k'}\big(a^{jk}\ell_j\big)_{k'}a^{j'k'}\big(v_{j'}\bar v_k+\bar v_{j'}v_k\big)-\sum_{j,k,j',k'}\big[\big(a^{j'k'}\ell_{j'}\big)_{k'}a^{jk}(v_j\bar v+\bar v_jv)\big]_k\\
&+\sum_{j,k,j',k'}\big[a^{jk}\big(a^{j'k'}\ell_{j'}\big)_{k'k}|v|^2\big]_j-\sum_{j,k,j',k'}\big[a^{jk}\big(a^{j'k'}\ell_{j'}\big)_{k'k}\big]_j|v|^2\\
&+\sum_{j,k,j',k'}a^{jk}\big(a^{j'k'}\ell_{j'}\big)_{k'}(v_j\bar v_k+\bar v_jv_k)+a\sum_{j,k}\big(a^{jk}\ell_j\ell_k|v|^2\big)_t-a\sum_{j,k}\big(a^{jk}\ell_j\ell_k\big)_t|v|^2\\
&-2\sum_{j,k,j',k'}\big(a^{jk}a^{j'k'}\ell_j\ell_{j'}\ell_{k'}|v|^2\big)_k+2\sum_{j,k,j',k'}a^{jk}\ell_j\big(a^{j'k'}\ell_{j'}\ell_{k'}\big)_k|v|^2,
\end{aligned}\tag{14}$$

where we have used the following facts

$$\begin{aligned}
&2v\bar v_t=\big(|v|^2\big)_t-(v_t\bar v-\bar v_tv),\\
&2\bar vv_k=\big(|v|^2\big)_k-(\bar v_kv-v_k\bar v),
\end{aligned}$$

and

$$2\sum_{j,k,j',k'} a^{jk}a^{j'k'}\ell_j(v_{j'}\bar{v}_{kk'}+\bar{v}_{j'}v_{kk'}) = \sum_{j,k,j',k'} \left[a^{jk}a^{j'k'}\ell_j(v_{j'}\bar{v}_{k'}+\bar{v}_{j'}v_{k'})\right]_k - \sum_{j,k,j',k'} \left(a^{jk}a^{j'k'}\ell_j\right)_k(v_{j'}\bar{v}_{k'}+\bar{v}_{j'}v_{k'}).$$

Finally, combining all '$\frac{\partial}{\partial t}$-terms' and all '$\frac{\partial}{\partial x_k}$-terms' in (14), and noting (13), we arrive at (5). □

Acknowledgements

This work is partially supported by the FANEDD of China (Project No: 200119), the NCET of China under grant NCET-04-0882, and the NSF of China under grants 10371084 and 10525105. The author gratefully acknowledges Professor Xu Zhang for his guidance and suggestions.

References

[1] L. Baudouin, J.P. Puel, Uniqueness and stability in an inverse problem for the Schrödinger equation, Inverse Problems 18 (2002) 1537–1554.
[2] A. Doubova, E. Fernández-Cara, M. Gouzález-Burgos, E. Zuazua, On the controllability of parabolic systems with a nonlinear term involving the state and the gradient, SIAM J. Control Optim. 41 (2002) 798–819.
[3] X. Fu, Null controllability for the parabolic equations with a complex principal part, Preprint.
[4] X. Fu, J. Yong, X. Zhang, Exact controllability for multidimensional semilinear hyperbolic equations, submitted for publication.
[5] A.V. Fursikov, O.Yu. Imanuvilov, Controllability of Evolution Equations, Lecture Notes Ser., vol. 34, Research Institute of Mathematics, Seoul National University, Seoul, Korea, 1994.
[6] I. Lasiecka, R. Triggiani, X. Zhang, Global uniqueness, observability and stabilization of nonconservative Schrödinger equations via pointwise Carleman estimates, Part I: $H^1(\Omega)$-estimates, J. Inverse Ill-Posed Probl. 12 (2004) 43–123.
[7] S. Tang, X. Zhang, Carleman inequality for the backward stochastic parabolic equations with general coefficients, C. R. Acad. Sci. Paris, Ser. I 339 (2004) 775–780.
[8] X. Zhang, Exact controllability of the semilinear plate equations, Asymptotic Anal. 27 (2001) 95–125.
[9] E. Zuazua, Some problems and results on the controllability of partial differential equations, in: Progr. in Math., vol. 169, Birkhäuser, Basel, Switzerland, 1998, pp. 276–311.
[10] E. Zuazua, Controllability of partial differential equations and its semi-discrete approximation, Discrete Contin. Dynam. Systems 8 (2002) 469–513.