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Numerical Analysis

Boundary value problems with nonhomogeneous Neumann conditions on a fractal boundary ☆

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Abstract

This note deals with some boundary value problems in a self-similar ramified domain of $\mathbb{R}^2$, with a fractal boundary. The partial differential equation is Laplace's equation, and there are nonhomogeneous generalized Neumann boundary conditions on the fractal boundary. We propose a multiscale strategy for approximating the restriction of the solutions to simple subdomains. This strategy is based on transparent boundary conditions and on a wavelet expansion of the Neumann datum. ***To cite this article: Y. Achdou, N. Tchou, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Problèmes aux limites avec des conditions de Neumann non homogènes sur une frontière fractale. On considère une classe de problèmes aux limites dans un domaine autosimilaire ramifié de $\mathbb{R}^2$ avec une frontière fractale. L'équation aux dérivées partielles est l'équation de Laplace et on impose des conditions de Neumann non homogènes sur la frontière fractale. On propose une stratégie multiéchelle pour calculer la solution dans des sous-domaines simples. Cette stratégie met en jeu un développement de la donnée de Neumann sur une base d'ondelettes et des conditions aux limites transparentes. ***Pour citer cet article : Y. Achdou, N. Tchou, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Dans cette Note, on considère une classe de problèmes aux limites avec l'équation de Laplace dans un domaine auto-similaire ramifié de $\mathbb{R}^2$.

Le domaine est construit à partir d'un motif de base Y^0 et de deux transformations affines F_1 et F_2 donnés par (1). Pour $n \geqslant 1$, soit $\mathcal{A}_n$ l'ensemble des 2^n applications de $\{1,\dots,n\}$ dans $\{1,2\}$. Pour $\sigma \in \mathcal{A}_n$, on définit l'application

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affine $\mathcal{M}_\sigma(F_1, F_2)$ par (2). Le domaine Ω^0 est construit dans (3) en assemblant les domaines Y^0 et $\mathcal{M}_\sigma(F_1, F_2)(Y^0)$, $\sigma \in \mathcal{A}_n$, $n \geqslant 1$, voir la figure dans [2]. Sa frontière est décrite par (4). On appelle Y^N le domaine obtenu en arrêtant la construction de Ω^0 à la $N+1$-ème étape, voir (5), et $\Omega^\sigma = \mathcal{M}_\sigma(F_1, F_2)(\Omega^0)$.

La partie Γ^∞ de $\partial\Omega^0$ (voir (4)) a la propriété suivante : $\forall x \in \Gamma^\infty$, $\forall r > 0$, $\Omega^0 \cap B(x, r)$ n'est pas un (ϵ, δ) domaine, voir e.g. [3,4], où $B(x, r)$ est la boule ouverte de centre x et de rayon r.

La Note [2] concernait des problèmes de Poisson dans Ω^0 avec une condition de Neumann homogène généralisée sur Γ^∞. On s'intéresse ici à des conditions de Neumann non homogènes sur Γ^∞. Pour écrire le problème aux limites en toute rigueur, on doit d'abord obtenir des résultats nouveaux sur les espaces de Sobolev sur Ω^0, en particulier des resultats de trace sur Γ^∞. Ces résultats sont donnés dans les Théorèmes 3.1, 3.2 et 3.3 ci dessous. On montre en particulier que pour p, $1 < p < 2$, l'opérateur de trace sur Γ^∞ défini sur $\mathcal{C}^\infty(\overline{\Omega^0})$, peut être étendu de manière unique en un opérateur surjectif de $W^{1,p}(\Omega^0)$ sur $W^{1-1/p,p}(\Gamma^\infty)$. Ceci permet de définir un opérateur de trace γ continu de $H^1(\Omega^0)$ dans $L^2(\Gamma^\infty)$, qui étend l'opérateur défini sur $\mathcal{C}^\infty(\overline{\Omega^0})$ et qui vérifie (6).

On peut alors considérer le problème (8), dont les solutions vérifient (9). Le Théorème 3.4 dit que (8) a une solution unique et donne un sens intrinsèque à (8), grâce au choix de γ. Il permet d'interpréter la condition aux limites sur Γ^∞ comme une condition de Neumann généralisée.

On souhaite calculer la restriction de la solution de (8) à Y^{n-1}, $n > 0$ fixé. Dans le cas où la donnée de Neumann sur Γ^∞ est nulle, on a montré dans [2] que ceci peut être fait en résolvant successivement $1 + \cdots + 2^{n-1}$ problèmes aux limites dans Y^0 avec des conditions aux limites non locales sur $F_1(\Gamma^0)$ et $F_2(\Gamma^0)$, appelées *conditions transparentes* et mettant en jeu un opérateur de Dirichlet–Neumann T défini par (13). De plus, on a montré dans [2] que T est l'unique point fixe d'une application $\mathbb{M}$, et peut être approché par des itérations de point de fixe de $\mathbb{M}$.

Dans cette Note, on s'intéresse au cas où la donnée de Neumann g sur Γ^∞ est non nulle. On considère d'abord le cas où g est une ondelette dans la base de Haar associée à une décomposition dyadique de Γ^∞. Dans ce cas, on montre dans le Theorème 4.1 et les Propositions 4.2, 4.3 que pour $n > 0$ fixé, les restrictions des solutions à Y^{n-1} peuvent être calculées en résolvant un nombre fini de problèmes dans Y^0 avec des conditions transparentes non homogènes sur $F_1(\Gamma^0)$ et $F_2(\Gamma^0)$. On montre dans le Théorème 4.4 que la contribution de Y^{n-1}, $n > 0$ fixé, à l'énergie de la solution décroît exponentiellement avec le niveau de l'ondelette g.

Pour une donnée de Neumann $g \in L^2(\Gamma^\infty)$ générale, on développe g sur la base des ondelettes de Haar, et on calcule la solution par superposition. La Proposition 4.5 donne une estimation d'erreur pour cette méthode. On a donc obtenu un algorithme de calcul de la restriction de la solution à Y^n, n fixé, et une estimation d'erreur. Les preuves des résultats ainsi que des calculs numériques sont contenus dans [1].

1. Introduction

This Note deals with some boundary value problems in a self-similar ramified domain of $\mathbb{R}^2$ with a fractal boundary: the domain Ω^0 whose construction is presented in Section 2 below, (see also the figure in [2]), has the following important property: there exists a straight line Γ^∞ (see (4) below), such that

(i) $\Gamma^\infty \subset \partial\Omega^0$,
(ii) $\forall x \in \Gamma^\infty$, $\forall r > 0$, $\Omega^0 \cap B(x, r)$ is not an (ϵ, δ) domain, see e.g. [3] or [4], where $B(x, r)$ is the two-dimensional ball with center x and radius r.

The Note [2] was devoted to Poisson problems with a generalized homogeneous Neumann condition on Γ^∞. Both Laplace and Helmholtz equation were considered. The present note deals with nonhomogeneous Neumann conditions on Γ^∞. The first goal is to give a rigorous functional setting allowing the study of variational formulations. In contrast with [2], trace results in Sobolev spaces are needed; to our knowledge, these results are new. The second goal is to propose a method for computing the solutions in simple subdomains of Ω^0. We use the Dirichlet–Neumann operator T introduced in [2]. When the Neumann data belongs to the Haar basis, we propose new boundary value problems in simple subdomains of Ω^0, with nonhomogeneous and nonlocal boundary conditions involving T. These can be called *transparent conditions*, as explained in [2]. For a general Neumann data g, the idea is to expand g on the Haar basis and use the linearity of the problem for deriving an expansion of the solution. The proofs of the results below are contained in [1]. The discrete counterparts of the methods have been successfully implemented in [1].

2. The geometry

We repeat the construction given in [2]. Let Y^0 and F_i, $i = 1, 2$, be respectively the T-shaped subset of $\mathbb{R}^2$ and the affine maps defined by:

$$Y^0 = \text{Interior}\big(\big([-1,1] \times [0,2]\big) \cup \big([-2,2] \times [2,3]\big)\big), \qquad F_i(x) = \left((-1)^i \frac{3}{2} + \frac{x_1}{2}, 3 + \frac{x_2}{2}\right), \quad i = 1, 2. \tag{1}$$

For $n \geqslant 1$, we call $\mathcal{A}_n$ the set containing all the 2^n mappings from $\{1, \ldots, n\}$ to $\{1, 2\}$. We define

$$\mathcal{M}_\sigma(F_1, F_2) = F_{\sigma(1)} \circ \ldots \circ F_{\sigma(n)} \quad \text{for } \sigma \in \mathcal{A}_n, \tag{2}$$

and the ramified open domain, see the figure in [1],

$$\Omega^0 = \text{Interior}\left(\overline{Y^0} \cup \left(\bigcup_{n=1}^{\infty} \bigcup_{\sigma \in \mathcal{A}_n} \mathcal{M}_\sigma(F_1, F_2)\big(\overline{Y^0}\big)\right)\right). \tag{3}$$

We split the boundary of Ω^0 into

$$\Gamma^0 = [-1,1] \times \{0\}, \quad \Gamma^\infty = [-3,3] \times \{6\}, \quad \text{and} \quad \Sigma^0 = \partial\Omega^0 \setminus \big(\Gamma^0 \cup \Gamma^\infty\big). \tag{4}$$

Let us define the subdomains Ω^σ ($\sigma \in \mathcal{A}_n$, $n > 0$), the polygonal open domain Y^N obtained by stopping the above construction at step $N + 1$,

$$\Omega^\sigma = \mathcal{M}_\sigma(F_1, F_2)\big(\Omega^0\big), \quad \text{and} \quad Y^N = \text{Interior}\left(\overline{Y^0} \cup \left(\bigcup_{n=1}^{N} \bigcup_{\sigma \in \mathcal{A}_n} \mathcal{M}_\sigma(F_1 F_2)\big(\overline{Y^0}\big)\right)\right). \tag{5}$$

We have $Y^N = \Omega^0 \setminus (\bigcup_{\sigma \in \mathcal{A}_{N+1}} \overline{\Omega^\sigma})$. We introduce the sets $\Gamma^\sigma = \mathcal{M}_\sigma(F_1, F_2)(\Gamma^0)$and $\Gamma^N = \bigcup_{\sigma \in \mathcal{A}_N} \Gamma^\sigma$.

3. A class of Poisson problems

3.1. Preliminary results

For a positive integer p and a real number $q \geqslant 1$, we define $W^{p,q}(\Omega^0)$ as the space of functions whose partial derivatives up to order p belong to $L^q(\Omega^0)$. We note $H^p(\Omega^0) = W^{p,2}(\Omega^0)$. Since Ω^0 is not an (ϵ, δ) domain, the extension and trace results in [3] and [4] cannot be applied. It is proved in [1] that there is no bounded extension operator from $H^1(\Omega^0)$ to $H^1(\mathbb{R}^2)$; the following extension result for Sobolev spaces is optimal:

Theorem 3.1. *There exists an extension operator $\mathcal{J}$ bounded from $W^{1,q}(\Omega^0)$ to $W^{1,q}(\mathbb{R}^2)$, for all q, $1 \leqslant q < 2$.*

Theorem 3.2. *For q, $1 \leqslant q < 2$, the space $\mathcal{C}^\infty(\overline{\Omega^0})$ is dense in $W^{1,q}(\Omega^0)$.*

Theorem 3.3 *(Trace). For q, $1 < q < 2$, the trace operator on Γ^∞ defined on $\mathcal{C}^\infty(\overline{\Omega^0})$ can be extended to a unique continuous operator from $W^{1,q}(\Omega^0)$ onto $W^{1-1/q,q}(\Gamma^\infty)$.*

As a consequence of Theorem 3.3, we can define a trace operator γ from $H^1(\Omega^0)$ to $L^2(\Gamma^\infty)$, which extends the trace operator defined on $\mathcal{C}^\infty(\overline{\Omega^0})$, (note that such an operator may not be unique since $\mathcal{C}^\infty(\overline{\Omega^0})$ is not dense in $H^1(\Omega^0)$), with the important property: for p, $4/3 < p < 2$,

$$\lim_{n \to \infty} \int_{\Gamma^n} u|_{\Gamma^n} g|_{\Gamma^n} = \frac{1}{3} \int_{\Gamma^\infty} \gamma(u) g|_{\Gamma^\infty}, \quad \forall u \in H^1(\Omega^0),\ \forall g \in W^{1,p}\big(\Omega^0\big). \tag{6}$$

We define

$$\mathcal{V}\big(\Omega^0\big) = \big\{v \in H^1\big(\Omega^0\big);\ v|_{\Gamma^0} = 0\big\}, \quad \text{and} \quad \mathcal{V}\big(Y^n\big) = \big\{v \in H^1\big(Y^n\big);\ v|_{\Gamma^0} = 0\big\}. \tag{7}$$

3.2. The boundary value problem

Take $g \in L^2(\Gamma^\infty)$ and $u \in H^{1/2}(\Gamma^0)$ and consider the variational problem: find $U(u,g) \in H^1(\Omega^0)$ s.t.

$$\big(U(u,g)\big)\big|_{\Gamma^0} = u, \quad \text{and} \quad \int\limits_{\Omega^0} \nabla\big(U(u,g)\big)\cdot\nabla v = \frac{1}{3}\int\limits_{\Gamma^\infty} g\gamma(v), \quad \forall v \in \mathcal{V}\big(\Omega^0\big). \tag{8}$$

If such a function $U(u,g)$ exists, then $(U(u,g))|_{\Gamma^0} = u$ and

$$\Delta\big(U(u,g)\big) = 0 \quad \text{in } \Omega^0, \qquad \frac{\partial(U(u,g))}{\partial n} = 0 \quad \text{on } \Sigma^0. \tag{9}$$

We shall discuss the boundary condition on Γ^∞ after Theorem 3.4 below:

Theorem 3.4. *For $g \in L^2(\Gamma^\infty)$ and $u \in H^{1/2}(\Gamma^0)$, problem* (8) *has a unique solution. Furthermore, if $g \in W^{1-1/p,p}(\Gamma^\infty)$, for some p, $4/3 < p < 2$, then $g \in L^2(\Gamma^\infty)$ and there exits $\tilde{g} \in W^{1,p}(\mathbb{R}^2)$ such that $g = \tilde{g}|_{\Gamma^\infty}$. Let $w_q \in H^1(Y^q)$ be the weak solution of*:

$$\Delta w_q = 0 \quad \textit{in } Y^q, \qquad w_q|_{\Gamma^0} = u, \qquad \frac{\partial w_q}{\partial n} = 0 \quad \textit{on } \partial Y_q \setminus \big(\Gamma^0 \cup \Gamma^{q+1}\big), \tag{10}$$

$$\frac{\partial w_q}{\partial n} = \tilde{g}|_{\Gamma^{q+1}} \quad \textit{on } \Gamma^{q+1}, \tag{11}$$

we have

$$\lim_{q\to\infty} \big\| \big(U(u,g)\big)\big|_{Y^q} - w_q \big\|_{H^1(Y^q)} = 0. \tag{12}$$

Theorem 3.4 says in particular that (8) has an intrinsic meaning if $g \in W^{1-1/p,p}(\Gamma^\infty)$, $4/3 < p < 2$. From the definition of w_q in (10) and (11), and the convergence result (12), we may say that $U(u,g)$ satisfies a generalized Neumann condition on Γ^∞, with datum g. Note that the factor $1/3$ in (8) has been chosen from property (6).

4. A method for computing the restriction of $U(u,g)$ to Y^n

As in [2], we use the notations $\mathcal{H}(u) = U(u,0)$, and T for the Dirichlet–Neumann operator from $H^{1/2}(\Gamma^0)$ to $(H^{1/2}(\Gamma^0))'$ defined by

$$Tu = -\frac{\partial \mathcal{H}(u)}{\partial x_2}\bigg|_{\Gamma^0}. \tag{13}$$

In [2], it is shown that if T is available, then, for a fixed integer $n > 0$, $\mathcal{H}(u)|_{Y^{n-1}}$ can be computed by solving successively $1+2+\cdots+2^{n-1}$ boundary value problems in Y^0, with nonlocal boundary conditions on Γ^1 involving T. Moreover, we have shown that T can be obtained as the limit of a sequence of operators constructed by a simple induction. Hereafter, we assume that T is available, therefore $\mathcal{H}(u)|_{Y^n}$, $n \geqslant 0$, can be computed by using the method presented in [2], and it is not restrictive to assume that $u = 0$. In what follows, we propose a method for computing $(U(0,g))|_{Y^{n-1}}$, (n is some fixed positive integer). We distinguish first the case when g belongs to the Haar basis associated to the dyadic decomposition of Γ^∞.

4.1. The case when the Neumann datum belongs to the Haar basis

The case when g is a Haar wavelet is particularly favorable, because one can make use of the self-similarity in the geometry in order to obtain boundary transparent conditions.

Let us call $e_F = U(0, 1_{\Gamma^\infty})$.

We introduce the linear operator B, bounded from $(H^{1/2}(\Gamma^0))'$ to $L^2(\Gamma^0)$, by: $Bz = -\frac{\partial w}{\partial x_2}|_{\Gamma^0}$, where $w \in \mathcal{V}(Y^0)$ is the unique weak solution to

$$\Delta w = 0 \quad \text{in } Y^0, \qquad \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial Y_0 \setminus \left(\Gamma^0 \cup \Gamma^1\right), \tag{14}$$

$$\left.\frac{\partial w}{\partial x_2}\right|_{F_i(\Gamma^0)} + 2\left(T(w|_{F_i(\Gamma^0)} \circ F_i)\right) \circ F_i^{-1} = -z \circ F_i^{-1}, \qquad i = 1, 2. \tag{15}$$

The self-similarity in the geometry and the scale-invariance of the equations in (9) are the fundamental ingredients for proving the following theorem:

Theorem 4.1. *The normal derivative y_F of e_F on Γ^0 belongs to $L^2(\Gamma^0)$ and is the unique solution to*

$$y_F = B y_F \quad \text{and} \quad \int_{\Gamma^0} y_F = -2. \tag{16}$$

For all $n \geqslant 1$, the restriction of e_F to Y^{n-1} can be computed by solving successively $1 + 2 + \cdots + 2^{n-1}$ boundary value problems in Y^0, as follows:

- *Loop: for $p = 0$ to $n - 1$,*
  - *Loop: for $\sigma \in \mathcal{A}_p$, (at this point, if $p > 0$, $e_F|_{\Gamma^\sigma}$ is known)*
    Solve the boundary value problem in Y^0: find $w \in H^1(\Omega^0)$ satisfying (14), and

$$\text{if } p = 0, \quad w|_{\Gamma^0} = 0, \quad \text{else} \quad w|_{\Gamma^0} = e_F|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1, F_2), \tag{17}$$

$$\left.\frac{\partial w}{\partial x_2}\right|_{F_i(\Gamma^0)} + 2\left(T(w|_{F_i(\Gamma^0)} \circ F_i)\right) \circ F_i^{-1} = -\frac{1}{2^p} y_F \circ F_i^{-1}, \quad i = 1, 2. \tag{18}$$

Set $e_F|_{Y^0} = w$ if $p = 0$, else set $e_F|_{\mathcal{M}_\sigma(F_1,F_2)(Y^0)} = w \circ (\mathcal{M}_\sigma(F_1, F_2))^{-1}$.

The knowledge of T, e_F and y_F permits the computation of $U(0, g)$ when g is a Haar wavelet on Γ^∞: call $g^0 = 1_{\{-3<x_1<0\}} - 1_{\{3>x_1>0\}}$ the Haar mother wavelet, and define $e^0 = U(0, g^0)$. By using the following result, one may compute $e^0|_{Y^n}$:

Proposition 4.2. *We have $e^0|_{Y^0} = w$, where $w \in \mathcal{V}(Y^0)$ satisfies (14) and*

$$\left.\frac{\partial w}{\partial x_2}\right|_{F_i(\Gamma^0)} + 2\left(T(w|_{F_i(\Gamma^0)} \circ F_i)\right) \circ F_i^{-1} = (-1)^i y_F \circ F_i^{-1}, \quad i = 1, 2. \tag{19}$$

Furthermore, for $i = 1, 2$,

$$e^0|_{F_i(\Omega_0)} = \frac{(-1)^{i+1}}{2} e_F \circ F_i^{-1} + \left(\mathcal{H}\left(e^0|_{F_i(\Gamma_0)} \circ F_i\right)\right) \circ F_i^{-1}. \tag{20}$$

For p a positive integer, take $\sigma \in \mathcal{A}_p$. Call g^σ the Haar wavelet on Γ^∞, defined by

$$g^\sigma|_{\mathcal{M}_\sigma(F_1,F_2)(\Gamma^\infty)} = g^0 \circ \mathcal{M}_\sigma^{-1}(F_1, F_2), \quad \text{and} \quad g^\sigma|_{\Gamma^\infty \setminus \mathcal{M}_\sigma(F_1,F_2)(\Gamma^\infty)} = 0, \tag{21}$$

and call $e^\sigma = U(0, g^\sigma)$, and y^σ (resp. y^0) the normal derivative of e^σ (resp. e^0) on Γ^0. The following result shows that (e^σ, y^σ) can be computed by induction:

Proposition 4.3. *The family (e^σ, y^σ) is defined by induction: assume that $\sigma = F_i \circ \eta$ for some $i \in \{1, 2\}$, $\eta \in \mathcal{A}_{p-1}$, $p > 1$. Then $e^\sigma|_{Y^0} = w$, where $w \in \mathcal{V}(Y^0)$ satisfies (14) and*

$$\left.\frac{\partial w}{\partial x_2}\right|_{F_i(\Gamma^0)} + 2\left(T(w|_{F_i(\Gamma^0)} \circ F_i)\right) \circ F_i^{-1} = -y^\eta \circ F_i^{-1}, \quad i = 1, 2. \tag{22}$$

Then, with $j = 1 - i$, $e^\sigma|_{\Omega^0 \setminus Y^0}$ is given by

$$e^\sigma|_{F_i(\Omega^0)} = \frac{1}{2} e^\eta \circ F_i^{-1} + \left(\mathcal{H}\left(e^\sigma|_{F_i(\Gamma_0)} \circ F_i\right)\right) \circ F_i^{-1}, \quad \text{and} \quad e^\sigma|_{F_j(\Omega^0)} = \left(\mathcal{H}\left(e^\sigma|_{F_j(\Gamma_0)} \circ F_j\right)\right) \circ F_j^{-1}. \tag{23}$$

If $\sigma = F_i$, $i = 1, 2$, then y^η (resp. e^η) must be replaced by y^0 (resp. e^0) in (22), (resp. (23)).

What follows indicates that for $n \geqslant 0$ fixed, $\|\nabla e^\sigma\|_{L^2(Y^n)}$, $\sigma \in \mathcal{A}_p$, decays exponentially as $p \to \infty$:

Theorem 4.4. *There exists two positive constants C and ρ, $0 < \rho < 1$, such that for all nonnegative integers n, p such that $0 \leqslant n < p - 1$, the function e^σ satisfies*

$$\|\nabla e^\sigma\|_{L^2(Y^n)} \leqslant C 2^{-n} \rho^{p-n}. \tag{24}$$

4.2. The general case

Consider now the case when g is a general function in $L^2(\Gamma^\infty)$. It is no longer possible to use the self-similarity in the geometry for deriving transparent boundary conditions for $U(0, g)$. The idea is different: one expands g on the Haar basis, and use the linearity of (8) with respect to g for obtaining an expansion of $U(0, g)$ in terms of e_F, e^0, and e^σ, $\sigma \in \mathcal{A}_p$, $p > 1$. Indeed, one can expand $g \in L^2(\Gamma^\infty)$ as follows:

$$g = \alpha_F 1_{\Gamma^\infty} + \alpha_0 g^0 + \sum_{p=1}^{\infty} \sum_{\sigma \in \mathcal{A}_p} \alpha_\sigma g^\sigma. \tag{25}$$

The following result, which a consequence of Theorem 4.4, says that $(U(0, g))|_{Y^n}$ can be expanded in terms of $e_F|_{Y^n}$, $e^0|_{Y^n}$, and $e^\sigma|_{Y^n}$, $\sigma \in \mathcal{A}_p$, $p \geqslant 1$. Moreover, a few terms in the expansion are enough to approximate $(U(0, g))|_{Y^n}$ with a good accuracy:

Proposition 4.5. *Assume that $g \in L^2(\Gamma^\infty)$ has the expansion* (25). *There exists a constant C* (*independent of g*) *such that for all integers n, P, with $0 \leqslant n < P - 1$,*

$$\left\| \left(U(0, g)\right)\big|_{Y^n} - \alpha_F e_F|_{Y^n} - \alpha_0 e^0\big|_{Y^n} - \sum_{p=1}^{P} \sum_{\sigma \in \mathcal{A}_p} \alpha_\sigma e^\sigma\big|_{Y^n} \right\|_{H^1(Y^n)} \leqslant C\sqrt{2^{-P}} \rho^{P-n} \|g\|_{L^2(\Gamma^\infty)}. \tag{26}$$

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