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Complex Analysis

A Note on the approximation of plurisubharmonic functions

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Abstract

Let $\Omega \Subset \mathbb{C}^n$ be a strongly hyperconvex domain and Ω_j be a decreasing sequence of hyperconvex domains such that $\Omega = (\bigcap \Omega_j)^{\circ}$. We show that every plurisubharmonic function $\varphi \in \mathcal{F}^a(\Omega)$ is a limit of an increasing sequence of functions $\varphi_j \in \mathcal{F}^a(\Omega_j)$. ***To cite this article: S. Benelkourchi, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Sur l'approximation des fonctions plurisousharmoniques. Soit $\Omega \Subset \mathbb{C}^n$ un domaine fortement hyperconvexe et Ω_j une suite décroissante de domaines hyperconvexes tel que $\Omega = (\bigcap \Omega_j)^{\circ}$. On prouve que toute fonction plurisousharmonique $\varphi \in \mathcal{F}^a(\Omega)$ est limite d'une suite croissante de fonctions $\varphi_j \in \mathcal{F}^a(\Omega_j)$. ***Pour citer cet article : S. Benelkourchi, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

The purpose of this Note is to answer to a question posed by U. Cegrell (cf. [3]) about the approximation of plurisubharmonic functions. We first recall some definitions (cf. [4]). Let $\Omega \Subset \mathbb{C}^n$ be a bounded hyperconvex domain, i.e. open, bounded, connected and there exists a continuous plurisubharmonic function $\rho : \Omega \to (-\infty, 0)$ such that the closure of the set $\{z \in \Omega:\ \rho(z) < c\}$ is compact in Ω for every $c \in (-\infty, 0)$. If moreover the function ρ is defined in a neighborhood Ω' of $\overline{\Omega}$ and $\Omega = \{\rho < 0\}$ then we say (see [7]) that Ω is strongly hyperconvex. Such function ρ is called an exhaustion function for Ω.

We denote by $\mathcal{E}_0(\Omega)$ the class of negative and bounded plurisubharmonic functions u on Ω which tends to 0 at the boundary of Ω and satisfy $\int_\Omega (dd^c u)^n < \infty$. Then denote by $\mathcal{F}(\Omega)$ the class of negative plurisubharmonic functions φ on Ω for which there exists a decreasing sequence (φ_j) of plurisubharmonic functions in $\mathcal{E}_0(\Omega)$ converging to φ on Ω such that

$$\sup_j \int_\Omega \left(dd^c \varphi_j\right)^n < +\infty.$$

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It is known (see [4]) that the complex Monge–Ampère operator is well defined for a function $\varphi \in \mathcal{F}(\Omega)$ as the weak-limit of the sequence of measures $(dd^c\varphi_j)^n$, where (φ_j) is any decreasing sequence of plurisubharmonic functions from the class $\mathcal{E}_0(\Omega)$ converging to φ and every negative plurisubharmonic function u on Ω, such that $(dd^c u)^n$ is well defined, is locally in $\mathcal{F}(\Omega)$; to every open, relatively compact $\omega \Subset \Omega$, there is a $u_\omega \in \mathcal{F}(\Omega)$ with $u_\omega = u$ on ω.

Finally we denote by $\mathcal{F}^a(\Omega)$ the subclass of functions u in $\mathcal{F}(\Omega)$ such that its complex Monge–Ampère measure $(dd^c u)^n$ vanishes on all pluripolar subsets of Ω.

Theorem 1.1. *Let $\Omega \Subset \mathbb{C}^n$ be a strongly hyperconvex domain, Ω_j be a decreasing sequence of hyperconvex domains containing Ω such that $\Omega = (\bigcap \Omega_j)^\circ$ and for each open set D containing $\overline{\Omega}$, there exists j such that $\Omega_j \subset D$. Suppose $\varphi \in \mathcal{F}^a(\Omega)$. Then there exists an increasing sequence of functions $\varphi_j \in \mathcal{F}^a(\Omega_j)$ such that* $\lim_{j\to+\infty} \varphi_j(z) = \varphi(z)$, $\forall z \in \Omega$.

2. Proof

We start by give a new characterization of the class $\mathcal{F}(\Omega)$ in terms of the relative Monge–Ampère Capacity of sublevel sets. We recall that if $E \Subset \Omega$ is a Borelean set, then the relative Monge–Ampère Capacity of the condenser (E, Ω) is defined by the formula (see [1])

$$\operatorname{cap}(E,\Omega) := \sup\left\{ \int_E \left(dd^c v\right)^n,\ v \in \mathrm{PSH}(\Omega);\ -1 \leqslant v \leqslant 0 \right\}.$$

Proposition 2.1. *A function $\varphi \in \mathcal{F}(\Omega)$ if and only if*

$$\limsup_{s\to 0} s^n \operatorname{cap}\bigl(\{z \in \Omega; \varphi \leqslant -s\}, \Omega\bigr) < \infty,$$

moreover, if $\varphi \in \mathcal{F}(\Omega)$ then

$$\int_\Omega \left(dd^c\varphi\right)^n = \lim_{s\to 0} s^n \operatorname{cap}\bigl(\{z \in \Omega; \varphi \leqslant -s\}, \Omega\bigr). \tag{1}$$

Proof. Let $\varphi \in \mathrm{PSH}^-(\Omega)$, from [4] there exist a sequence of functions $\varphi_j \in \mathcal{E}_0(\Omega)$ such that $\varphi_j \searrow \varphi$ in Ω, and then we have (cf. [2,1,6]), for every j

$$s^n \operatorname{cap}\bigl(\{\varphi_j \leqslant -s\}, \Omega\bigr) \leqslant \int_\Omega \left(dd^c\varphi_j\right)^n, \quad \forall s > 0. \tag{2}$$

If $\varphi \in \mathcal{F}(\Omega)$ then $\sup_j \int_\Omega (dd^c\varphi_j)^n < +\infty$, and therefore $\limsup_{s\to 0} s^n \operatorname{cap}(\{z \in \Omega;\ \varphi \leqslant -s\}, \Omega) < \infty$.

Now, assume that $\limsup_{s\to 0} s^n \operatorname{cap}(\{z \in \Omega; \varphi \leqslant -s\}, \Omega) < \infty$, from [5] we have, for every j

$$\int_{\{\varphi_j \leqslant -s\}} \left(dd^c\varphi_j\right)^n \leqslant s^n \operatorname{cap}\bigl(\{\varphi_j \leqslant -s\}, \Omega\bigr), \tag{3}$$

then

$$\int_\Omega \left(dd^c\varphi_j\right)^n \leqslant \limsup_{s\to 0} s^n \operatorname{cap}\bigl(\{\varphi \leqslant -s\}, \Omega\bigr) < \infty,$$

which imply that $\varphi \in \mathcal{F}$.

For the second claim, let $\varphi \in \mathcal{F}$, from Corollary 3.4 and Proposition 5.1 in [4] and the inequality (3) we get

$$\int_\Omega \left(dd^c\varphi\right)^n \leqslant \liminf_{s\to 0} s^n \operatorname{cap}\bigl(\{z \in \Omega; \varphi \leqslant -s\}, \Omega\bigr).$$

Using the inequalities (2) once more, we get

$$\limsup_{s\to 0} s^n \operatorname{cap}\big(\{\varphi \leqslant -s\}, \Omega\big) \leqslant \int_\Omega \big(dd^c\varphi\big)^n,$$

hence (1) is proved. □

Lemma 2.2. *Let Ω and Ω_j be as in the preceding theorem. Let K be a compact subset of Ω. Then* $\operatorname{cap}(K, \Omega_j) \to \operatorname{cap}(K, \Omega)$*, as $j \to \infty$.*

Proof. We may assume that K is a regular compact. Let denote h_{K,Ω_j} the relative extremal function of the condenser (K, Ω_j). It's clear that $(h_{K,\Omega_j})_j$ is increasing sequence of functions on Ω, then $\lim_{j\to\infty} h_{K,\Omega_j} \leqslant h_{K,\Omega}$ on Ω. We claim that $\lim_{j\to\infty} h_{K,\Omega_j} = h_{K,\Omega}$.

Let $\varepsilon > 0$ and $0 < c < \varepsilon$, define

$$v = \begin{cases} \max(h_{K,\Omega} - \varepsilon, \rho - c) & \text{on } \Omega, \\ \rho - c & \text{on } \Omega' \backslash \Omega. \end{cases}$$

For j an integer big enough ($\Omega_j \subset \{z \in \Omega';\ \rho(z) < c\}$), the function $v \in \mathrm{PSH}^-(\Omega_j)$ and $v_{/K} \leqslant -1$, thus $h_{K,\Omega} - \varepsilon \leqslant h_{K,\Omega_j}$ for all j bigger than some an integer $j_0 = j_0(\varepsilon)$.
Letting j tend to ∞ and ε tend to 0, we get $h_{K,\Omega} \leqslant \lim_{j\to\infty} h_{K,\Omega_j}$ and the claim is proved.

It follows by [1] that $(dd^c h_{K,\Omega_j})^n$ converges weakly to $(dd^c h_{K,\Omega})^n$, thus

$$\limsup_{j\to\infty} \int_K \big(dd^c h_{K,\Omega_j}\big)^n \leqslant \int_K \big(dd^c h_{K,\Omega}\big)^n = \int_\Omega \big(dd^c h_{K,\Omega}\big)^n \leqslant \liminf_{j\to\infty} \int_\Omega \big(dd^c h_{K,\Omega_j}\big)^n,$$

and the lemma is proved. □

Now we are ready to prove Theorem 1.1. Fix an integer j and observe that the measure $\mu = \chi_\Omega (dd^c\varphi)^n$ put no mass on the pluripolar subsets of Ω_j and $\mu(\Omega_j) < \infty$, then by [4], there exist a unique function $\varphi_j \in \mathcal{F}^a(\Omega_j)$ such that

$$\big(dd^c\varphi_j\big)^n = \chi_\Omega \big(dd^c\varphi\big)^n.$$

It follows from Theorem 5.15 in [4] that the sequence of functions φ_j is increasing.

Denote $\tilde\varphi = (\lim \varphi_j)^*$, then we have $\tilde\varphi$ is negative and plurisubharmonic on Ω. We claim that $\tilde\varphi \in \mathcal{F}(\Omega)$.

Indeed, from the inequality (2), we have

$$s^n \operatorname{cap}\big(\{\varphi_j \leqslant -s\}, \Omega_j\big) \leqslant \int_{\Omega_j} \big(dd^c\varphi_j\big)^n,$$

and since $\varphi_j \leqslant \tilde\varphi$, it follows that

$$s^n \operatorname{cap}\big(\{\tilde\varphi \leqslant -s\}, \Omega_j\big) \leqslant \int_\Omega \big(dd^c\varphi\big)^n, \quad \forall j,$$

thus, by the preceding lemma

$$s^n \operatorname{cap}\big(\{\tilde\varphi \leqslant -s\}, \Omega\big) \leqslant \int_\Omega \big(dd^c\varphi\big)^n < \infty,$$

which implies by the preceding proposition that $\tilde\varphi \in \mathcal{F}(\Omega)$.

Now the sequence φ_j increases to $\tilde\varphi \in \mathcal{F}(\Omega)$ on Ω and $(dd^c\varphi_j)^n = \chi_\Omega (dd^c\varphi)^n$, it follows by the continuity of the complex Monge–Ampère operator under increasing sequence that $(dd^c\tilde\varphi)^n = (dd^c\varphi)^n$. Since the comparison principle still holds in $\mathcal{F}^a(\Omega)$ (see [4]) then $\tilde\varphi = \varphi$ which proves the theorem.

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