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Motives and modules over motivic cohomology

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Abstract

In this Note we summarize the main results and techniques in our homotopical algebraic approach to motives. A major part of this work relies on highly structured models for motivic stable homotopy theory. For any noetherian and separated base scheme of finite Krull dimension these frameworks give rise to a homotopy theoretic meaningful study of modules over motivic cohomology. When the base scheme is Spec(k), for k a field of characteristic zero, the corresponding homotopy category is equivalent to Voevodsky's big category of motives. *To cite this article: O. Röndigs, P.A. Østvær, C. R. Acad. Sci. Paris, Ser. I 342 (2006).* © 2006 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Résumé

Motifs et modules sur la cohomologie motivique. Dans cette Note, nous présentons nos résultats principaux et les techniques utilisées dans notre étude des motifs, qui est basée sur la théorie d'homotopie. Une partie importante de ce travail utilise des modèles hautement structurés pour la théorie d'homotopie stable motivique. Pour tout schéma de base noethérien, séparé et de dimension de Krull finie, ces outils permettent l'étude de la théorie d'homotopie des modules sur la cohomologie motivique. Lorsque le schéma de base est Spec(k), pour k un corps de caractéristique zéro, la catégorie homotopique obtenue est équivalente à la grande catégorie des motifs introduite par Voevodsky. *Pour citer cet article : O. Röndigs, P.A. Østvær, C. R. Acad. Sci. Paris, Ser. I 342 (2006).* © 2006 Académie des sciences. Published by Elsevier SAS. All rights reserved.

1. Introduction

We introduce a category of motives for noetherian and separated base schemes of finite Krull dimension. Its construction is based on highly structured models for the motivic stable homotopy category [2,6]. Further details are presented in [9] and [10]. For fields of characteristic zero, Voevodsky's big category of motives is shown to be equivalent to the homotopy category of a homotopy theoretic category of motives. Throughout this Note we make use of homotopical algebra.

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2. Main results

2.1. The unstable theory

Let Sm be the smooth Nisnevich site of a noetherian and separated scheme S of finite Krull dimension. There is a linearization functor from Sm to the Suslin–Voevodsky category Cor of finite correspondences of S [13,16]. A pointed motivic space is a pointed simplicial presheaf on Sm. A motivic space with transfers is a Z-linear functor from Cor^{op} to simplicial abelian groups. Let M and M^{tr} denote the corresponding functor categories. In [9] it is noted that M^{tr} has a tensor product \otimes^{tr} . Any scheme $U \in$ Sm defines a representable pointed motivic space U_+ – by adding a disjoint basepoint – and a representable motivic space with transfers U^{tr} . Let Δ^n denote the standard *n*-simplex. There is an evident forgetful functor $u: \mathbf{M}^{tr} \to \mathbf{M}$ induced by the graph functor Sm \to Cor. Its left adjoint Z^{tr} – the transfer functor – is determined by

$$\mathbb{Z}^{\mathrm{tr}}(U \times \Delta^{n})_{+} = U^{\mathrm{tr}} \otimes^{\mathrm{tr}} \mathbb{Z}[\Delta^{n}].$$
⁽¹⁾

In [9], we use the motivic model structure on M [2] to derive what we call the motivic model on M^{tr} .

Theorem 1. There exists a monoidal and simplicial model structure for motivic spaces with transfers such that u detects and preserves motivic weak equivalences and fibrations.

The category \mathbf{Pre}^{tr} of presheaves with transfers consists of \mathbb{Z} -linear functors from Cor^{op} to abelian groups. For fields, this category was introduced in [14]. Denote by

$$\mathbf{M}^{\mathrm{tr}} \Longleftrightarrow \mathrm{Ch}_{+}(\mathbf{Pre}^{\mathrm{tr}}) \tag{2}$$

the Dold–Kan equivalence between motivic spaces with transfers and connective (positively graded) chain complexes of presheaves with transfers. We may transport the motivic model on \mathbf{M}^{tr} to a motivic model on $\mathbf{Ch}_{+}(\mathbf{Pre}^{tr})$, turning (2) into a Quillen equivalence. Moreover, this model structure can be extended without much trouble to non-connected (unbounded) chain complexes of presheaves with transfers.

2.2. The stable theory

We are interested in stabilizing the motivic model structures using symmetric spectra [3]. Let S_s^1 denote the simplicial circle. For motivic symmetric spectra **MSS** we suspend with respect to the cofibrant motivic space $T = S_s^1 \wedge \mathbb{G}$ where \mathbb{G} is a cofibrant replacement for $(\mathbb{A}^1 \setminus \{0\}, 1)$. Applying the transfer functor to T, we obtain motivic symmetric spectra with transfers, denoted by **MSS**^{tr}. Let \mathbb{G}_m^{tr} be $\mathbb{Z}^{tr}(\mathbb{A}^1 \setminus \{0\}, 1)$, and **ChSS**_{+,\mathbb{G}_m^{tr}[1]}^{tr} the symmetric spectra of connective chain complexes of presheaves with transfers with respect to \mathbb{G}_m^{tr} shifted one degree. Similarly, we define **ChSS**_{+,\mathbb{P}^1}^{tr} by suspending with respect to $\mathbb{Z}^{tr}(\mathbb{P}^1, 1)$. Although the projective line $(\mathbb{P}^1, 1)$ pointed by $1: S \to \mathbb{P}^1$ is not cofibrant in the motivic model structure, $\mathbb{Z}^{tr}(\mathbb{P}^1, 1)$ is a cofibrant motivic space with transfers; this is relevant when stabilizing the motivic model. By [9], there is a zig–zag of strict symmetric monoidal Quillen equivalences between **MSS**^{tr} and **MSS**_{\mathbb{P}^1}^{tr}. There exists a same type of zig–zag between **ChSS**_{+,\mathbb{P}^1}^{tr} and **ChSS**_{+,\mathbb{G}_m^{tr}[1]}^{tr}. Since $\mathbb{Z}^{tr}(\mathbb{P}^1, 1)$ is cofibrant and discrete, the Dold–Kan equivalence (2) induces a lax symmetric monoidal Quillen equivalence between **MSS**_{\mathbb{P}^1}^{tr} and **ChSS**_{+,\mathbb{P}^1}^{tr} [9]. The stable homotopy theoretic forerunner of this result was proved in [12].

Let $M\mathbb{Z}$ denote motivic cohomology, considered as an object of **MSS** (see the next section). In [10] we compare $M\mathbb{Z}$ -mod with **MSS**^{tr}. Since the monoid axiom in [11] holds for motivic symmetric spectra [6], the module category over motivic cohomology acquires a model structure. A map between $M\mathbb{Z}$ -modules is a weak equivalence if the underlying map of motivic symmetric spectra is a stable weak equivalence.

Theorem 2. The model categories $M\mathbb{Z}$ -mod and MSS^{tr} are Quillen equivalent when the base scheme is Spec(k), for *k* a field of characteristic zero.

Using the naturally induced Quillen equivalence between $\mathbf{ChSS}_{+,\mathbb{G}_m^{tr}[1]}^{tr}$ and $\mathbf{ChSS}_{\mathbb{G}_m^{tr}[1]}^{tr}$ – obtained from nonconnected chain complexes of presheaves with transfers – it is now relatively straightforward to show the equivalence between the homotopy category Ho(M \mathbb{Z} -mod) and Voevodsky's big category of motives of \mathbb{G}_m^{tr} -spectra of nonconnected chain complexes of Nisnevich sheaves with transfers having homotopy invariant cohomology sheaves. The equivalence preserves both the monoidal and triangulated structures.

3. Outline of proofs

Theorem 1 is a 'projective' analogue for motivic spaces with transfers of the Morel–Voevodsky model structure for motivic unstable homotopy theory [7]. Its construction is accomplished in three steps. First, one defines a model structure on \mathbf{M}^{tr} by defining weak equivalences and cofibrations schemewise [2]. Second, following [5], one introduces the local projective model structure in which weak equivalences between motivic spaces with transfers can be tested on local Hensel rings. Third, following the guiding principle of inverting the affine line \mathbb{A}^1 , the motivic projective model structure is constructed by localizing the local projective model structure along the lines of [7]. An advantage of working 'projectively' is that the motivic model structure is monoidal. In addition, fibrancy involves the standard Nisnevich descent condition: \mathcal{Z} is fibrant in the motivic model structure if and only if the canonical map $\mathbb{A}^1 \to S$ induces a schemewise weak equivalence $\mathcal{Z}(-) \to \mathcal{Z}(\mathbb{A}^1 \times -)$ and applying \mathcal{Z} to any Nisnevich distinguished square (where *i* is an open embedding, *p* is an étale map, and $p^{-1}(X \setminus i(U)) \to X \setminus i(U)$ induces an isomorphism of reduced schemes)

yields a homotopy pullback square of simplicial abelian groups.

The proof of Theorem 2 makes use of recent developments of our understanding of the motivic stable homotopy category. In particular, we work with motivic functors as a model for motivic stable homotopy theory [2], and we employ Spanier–Whitehead duality (outlined in [15] and established in [4] and [8]).

Denote finitely presentable motivic spaces by fM. Motivic cohomology is the composite motivic functor

$$\mathbf{M}\mathbb{Z}: \mathbf{f}\mathbf{M} \subseteq \mathbf{M} \xrightarrow{\mathbb{Z}^{u}} \mathbf{M}^{\mathrm{tr}} \to \mathbf{M}.$$

$$\tag{4}$$

The transfer functor $\mathbb{Z}^{tr} : \mathbf{M} \to \mathbf{M}^{tr}$ is strict symmetric monoidal and u is lax symmetric monoidal. Thus, evaluating motivic cohomology on the sequence of finitely presentable motivic spaces $S_+, T, T^{\wedge 2}, \ldots$ yields a commutative motivic symmetric ring spectrum – also denoted by \mathbb{MZ} – which is weakly equivalent to Voevodsky's motivic Eilenberg–MacLane spectrum [2]. An \mathbb{MZ} -module is a motivic symmetric spectrum E with an action $\mathbb{MZ} \wedge E \to E$ satisfying the usual module conditions. Let \mathbb{MZ} -mod denote the category of \mathbb{MZ} -modules. As noted above, \mathbb{MZ} -mod acquires a stable model structure with weak equivalences and fibrations defined on underlying motivic symmetric spectra. It also follows that the triangulated homotopy category $\mathrm{Ho}(\mathbb{MZ}$ -mod) is generated by free \mathbb{MZ} -modules of shifted motivic symmetric suspension spectra of representable motivic spaces. Since every motivic symmetric spectrum with transfers has an evident \mathbb{MZ} -module structure, there exists a functor $\Psi : \mathbf{MSS}^{\mathrm{tr}} \to \mathbb{MZ}$ -mod preserving all limits and filtered colimits. The latter implies there exists a left adjoint functor $\Phi : \mathbb{MZ}$ -mod $\to \mathbf{MSS}^{\mathrm{tr}}$. Theorem 1 implies that (Φ, Ψ) is a Quillen adjoint pair such that Ψ detects weak equivalences of fibrant modules. Our goal is to show it is a Quillen equivalence. Equivalently, the unit map $\mathbb{MZ} \wedge U_+ \to \Psi \Phi (\mathbb{MZ} \wedge U_+)$ is a weak equivalence of motivic symmetric spectra for every smooth quasi-projective S-scheme U [10].

We shall analyze the unit map using the category **MF** of motivic functors $\mathbf{fM} \to \mathbf{M}$ introduced in [2]. If $X, Y \in \mathbf{MF}$, the enriched left Kan extension along the 'sphere spectrum' or unit $\mathbb{I}: \mathbf{fM} \subseteq \mathbf{M}$ extends X to an **M**-functor $\mathbb{I}_*X: \mathbf{M} \to \mathbf{M}$. Set $X \circ Y := \mathbb{I}_*X \circ Y$, where \circ denotes composition of functors. There is a natural assembly map $X \wedge Y \to X \circ Y$ where \wedge is the symmetric monoidal product in **MF**. It is an isomorphism when Y is representable [2]. If X and Y are represented by $A, B \in \mathbf{fM}$ respectively, then the assembly map is the natural adjointness isomorphism between $\mathbf{M}(A, -) \wedge \mathbf{M}(B, -)$ – which is $\mathbf{M}(A \wedge B, -)$ by definition – and $\mathbf{M}(A, \mathbf{M}(B, -))$. A routine check shows the evaluation of $\mathbb{MZ} \wedge (- \wedge B) \to \mathbb{MZ} \circ (- \wedge B)$ at the sequence $S_+, T, T^{\wedge 2}, \ldots$ coincides with the unit map $\mathbb{MZ} \wedge B \to \Psi \Phi(\mathbb{MZ} \wedge B)$. This reduces the proof of Theorem 2 to a question concerning assembly maps of motivic functors.

By [2], the structure of **MF** induces a monoidal product \star , unit I, and internal hom objects [-, -] on the motivic stable homotopy category. Then Z is dualizable if there is a canonical isomorphism

$$[Z,\mathbb{I}] \star Z \to [Z,Z]. \tag{5}$$

When S = Spec(k), for k a field of characteristic zero, and Z is a motivic functor represented by a smooth quasiprojective k-scheme, then Z is dualizable by [4,8]. Suppose B is a cofibrant finitely presentable motivic space, and the motivic functor $- \wedge B$ is dualizable. If X preserves weak equivalences between cofibrant finitely presentable motivic spaces, then the assembly map $X \wedge (- \wedge B) \rightarrow X \circ (- \wedge B)$ is a weak equivalence [10]. We note the forgetful functor $\mathbf{M}^{\text{tr}} \rightarrow \mathbf{M}$ preserves weak equivalences and fibrations. Thus, its left adjoint preserves weak equivalences between cofibrant motivic spaces. It follows that motivic cohomology, defined by (4), preserves weak equivalences between cofibrant finitely presentable motivic spaces. As a consequence of the theory developed, this suffices to conclude the proof of Theorem 2.

In [10], working with rational coefficients and using alterations in the sense of de Jong [1], we obtain for perfect fields a Quillen equivalence between MQ-mod and $MSS_{\mathbb{Q}}^{tr}$.

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