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Partial Differential Equations

# The sector of analyticity of nonsymmetric submarkovian semigroups generated by elliptic operators

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**Abstract**

We prove that a lower bound for the angle $\theta_p$ of the sector of analyticity of not necessarily symmetric submarkovian semigroups generated by second order elliptic operators in divergence form or by Ornstein–Uhlenbeck in $L^p_\mu$ is given by $\cot\theta_p = \sqrt{(p-2)^2 + p^2(\cot\theta_2)^2}/(2\sqrt{p-1})$. If the semigroup is symmetric then we recover known results. In general, this lower bound is optimal. ***To cite this article: R. Chill et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***


**Résumé**

**Le secteur d'analyticité de semi-groupes sous-markoviens non-symétriques engendrés par des opérateurs elliptiques.** Nous prouvons qu'une borne inférieure de l'angle $\theta_p$ du secteur d'analyticité de semi-groupes sous-markoviens non nécessairement symétriques qui sont engendrés par des opérateurs elliptiques sous forme divergencielle ou par des opérateurs de Ornstein–Uhlenbeck dans $L^p_\mu$ est donnée par la formule $\cot\theta_p = \sqrt{(p-2)^2 + p^2(\cot\theta_2)^2}/(2\sqrt{p-1})$. Si le semi-groupe est symétrique on retrouve alors des résultats connus. En général, cette borne inférieure est optimale. ***Pour citer cet article : R. Chill et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***


## Version française abrégée

Soit $\Omega \subset \mathbb{R}^N$ ouvert, et soit $S \in L^\infty(\Omega; \mathbb{R}^{N\times N})$ uniformément elliptique et uniformément sectoriel (voir (1)). Soit $m$ une fonction positive sur $\Omega$ telle que $m, m^{-1} \in L^\infty_{\mathrm{loc}}(\Omega)$, et soit $\mathrm{d}\mu = m\,\mathrm{d}\lambda$, où $\lambda$ est la mesure de Lebesgue. Soient $L^p_\mu := L^p(\Omega; \mathrm{d}\mu)$ et $H^1_\mu := H^1(\Omega; \mathrm{d}\mu)$ les espaces de Lebesgue et l'espace de Sobolev avec poids. Alors l'opérateur $A_2$ dans $L^2_\mu$ défini par

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$$D(A_2) := \left\{ u \in L^2_\mu \colon \exists v \in L^2_\mu \text{ s.t. } \forall \varphi \in H^1_\mu \colon \int_\Omega \langle S(x)\nabla u, \nabla v \rangle \, d\mu = \langle v, \varphi \rangle_{L^2_\mu} \right\}, \quad A_2 u := v,$$

qui est associé à la forme $(a, H^1_\mu)$ donnée par

$$a(u, v) := \int_\Omega \langle S(x)\nabla u, \nabla v \rangle \, d\mu, \quad u, v \in H^1_\mu,$$

est le générateur négatif d'un semi-groupe sous-markovien $(e^{-tA_2})_{t \geqslant 0}$ sur $L^2_\mu$, c.à.d. $e^{tA_2}$ est un opérateur positif contractant dans $L^2_\mu$ et dans $L^\infty$. On ne suppose pas que $A_2$ est auto-adjoint.

L'opérateur $A_2$ et la forme $a$ étant sectoriels, ce semi-groupe se prolonge en un semi-groupe analytique de contractions sur le secteur $\Sigma_{\theta_2} := \{z \in \mathbb{C} \colon |\arg z| < \theta_2\}$, où l'angle $\theta_2$ est donné par $\cot\theta_2 = c_2$. Par contractivité dans $L^\infty$, le semi-groupe s'extrapole dans $L^p_\mu$, $2 \leqslant p \leqslant \infty$, et par dualité aussi dans $L^p_\mu$, $1 < p \leqslant 2$. Le générateur négatif dans $L^p_\mu$ sera noté $A_p$. On démontre le résultat suivant :

**Théorème 0.1.** *Pour tout $1 < p < \infty$, le semi-groupe $(e^{-tA_p})_{t \geqslant 0}$ sur $L^p_\mu$ se prolonge en un semi-groupe analytique de contractions sur le secteur $\Sigma_{\theta_p}$, où*

$$\cot\theta_p = \frac{\sqrt{(p-2)^2 + p^2 c_2^2}}{2\sqrt{p-1}}.$$

**Exemple 1.** Le Théorème 0.1 s'applique aux opérateurs elliptiques avec des coefficients mesurables, réels et bornés :

$$Au = -\operatorname{div} S(x)\nabla u \quad \text{dans } \Omega.$$

Il suffit de poser $\mu = \lambda$.

**Exemple 2.** Le Théorème 0.1 s'applique également aux opérateurs d'Ornstein–Uhlenbeck donnés par les formules (4) et (5), si $\mu$ est la mesure invariante associée.

**Remarque 1.** L'estimation de l'angle d'analyticité obtenue dans le Théorème 0.1 est en général meilleure que celle obtenue par le Théorème d'interpolation de Stein, [10]. Cet angle serait $\theta_2(1 - |\frac{2}{p} - 1|)$.

**Remarque 2.** Dans le cas où les $S(x)$ sont symétriques, et donc l'opérateur $A_2$ est auto-adjoint, le Théorème 0.1 a été démontré dans [1,3,8]. Pour des semi-groupes sous-markoviens symétriques généraux, voir [5,6].

**Remarque 3.** Le choix du domaine de l'opérateur $A_2$ correspond à des conditions au bord de type Neumann. La démonstration du Théorème 0.1 montre que le domaine $H^1_\mu$ de la forme $a$ peut être remplacé par l'espace $H^1_{\mu,0}$ (la fermeture de $C_c^\infty(\Omega)$ dans $H^1_\mu$) ce qui correspond alors à des conditions au bord de Dirichlet si $m$ est non-dégénérée.

**Remarque 4.** Le semi-groupe $(e^{-tA_p})_{t \geqslant 0}$ peut évidemment être analytique dans un secteur plus large que celui donné par l'angle $\theta_p$. Il suffit de considérer $S$ non-symétrique mais constant et $\mu = \lambda$ la mesure de Lebesgue. Alors on a $c_2 > 0$, mais l'opérateur $A_2$ est auto-adjoint et il est connu que le semi-groupe $(e^{-tA_p})_{t \geqslant 0}$ est analytique dans le secteur $\Sigma_{\theta_p}$ avec $\cot\theta_p = |p-2|/(2\sqrt{p-1})$, ce qui est le cas de notre $\theta_p$ lorsque $c_2 = 0$. En général, l'angle $\theta_p$ obtenu dans le Théorème 0.1 est optimal ; voir [4,11] dans le cas symétrique. Dans le cas non-symétrique, il suit de [2, Theorem 2] que si $A$ est l'opérateur de Ornstein–Uhlenbeck défini en (4) sur l'espace $L^p_\mu$ ($\mu$ la mesure invariante associée), alors pour tout $p \in (1, \infty)$ l'angle d'analyticité $\theta_p$ du Théorème 0.1 est optimal. Plus précisement : si $(e^{-tA_p})_{t \geqslant 0}$ se prolonge en un semi-groupe analytique sur un secteur $\Sigma_\theta$ (ce semi-groupe n'est a priori pas un semi-groupe de contractions), alors $\theta \leqslant \theta_p$.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be open, and let $S \in L^\infty(\Omega; \mathbb{R}^{N\times N})$ be uniformly elliptic, i.e. $\mathrm{Re}\langle S(x)\xi, \xi\rangle \geqslant \eta|\xi|^2$ for every $x \in \Omega$, $\xi \in \mathbb{C}^N$ and some $\eta > 0$. Here $\langle\cdot,\cdot\rangle$ denotes the usual hermitian product in $\mathbb{C}^N$. Assume that $S$ is in addition uniformly sectorial, i.e., there exists a constant $c_2 \geqslant 0$ such that

$$\left|\mathrm{Im}\big\langle S(x)\xi, \xi\big\rangle\right| \leqslant c_2 \,\mathrm{Re}\big\langle S(x)\xi, \xi\big\rangle \quad \text{for all } x \in \Omega,\ \xi \in \mathbb{C}^N. \tag{1}$$

Let $m$ be a positive function in $\Omega$ such that $m, m^{-1} \in L^\infty_{\mathrm{loc}}(\Omega)$ and define the Borel measure $\mathrm{d}\mu = m\,\mathrm{d}\lambda$, where $\lambda$ is the Lebesgue measure on $\Omega$. Let us introduce the weighted spaces $L^p_\mu = L^p(\Omega; \mathrm{d}\mu)$ and $H^1_\mu = \{u \in H^1_{\mathrm{loc}}(\Omega)$: $u, \nabla u \in L^2_\mu\}$. The operator $A_2$ on $L^2_\mu$ defined by

$$D(A_2) := \left\{u \in L^2_\mu\colon \exists v \in L^2_\mu \text{ s.t. } \forall \varphi \in H^1_\mu\colon \int_\Omega \big\langle S(x)\nabla u, \nabla v\big\rangle \,\mathrm{d}\mu = \langle v, \varphi\rangle_{L^2_\mu}\right\}, \quad A_2 u := v, \tag{2}$$

which is associated with the form $(a, H^1_\mu)$

$$a(u, v) := \int_\Omega \big\langle S(x)\nabla u, \nabla v\big\rangle \,\mathrm{d}\mu, \quad u, v \in H^1_\mu, \tag{3}$$

is the negative generator of a submarkovian semigroup $(\mathrm{e}^{-tA_2})_{t\geqslant 0}$, i.e., $\mathrm{e}^{-tA_2}$ is a positive contraction which is also $L^\infty$-contractive [7,9]. Let us stress that we are not assuming that $A_2$ is self-adjoint.

By (1), the form $a$ and the operator $A_2$ are sectorial, and the semigroup $(\mathrm{e}^{-tA_2})_{t\geqslant 0}$ extends to an analytic contraction semigroup on the sector $\Sigma_{\theta_2} := \{z \in \mathbb{C}\colon |\arg z| < \theta_2\}$, where the angle $\theta_2$ is determined by the constant $c_2$ in (1) through the equation $\cot\theta_2 = c_2$.

Moreover, by $L^\infty$ contractivity, the semigroup $(\mathrm{e}^{-tA_2})_{t\geqslant 0}$ extrapolates on all $L^p_\mu$, $2 \leqslant p \leqslant \infty$. Since $A_2^*$ is the operator associated with the matrix $S^*$, it is also the negative generator of a submarkovian semigroup and therefore, by duality, the semigroup $(\mathrm{e}^{-tA_2})_{t\geqslant 0}$ extrapolates on all $L^p_\mu$, $1 < p \leqslant \infty$. The negative generator on $L^p_\mu$ will be denoted by $A_p$. We prove the following theorem:

**Theorem 1.1.** *For every $1 < p < \infty$, the semigroup $(\mathrm{e}^{-tA_p})_{t\geqslant 0}$ on $L^p_\mu$ extends to an analytic semigroup of contractions on the sector $\Sigma_{\theta_p}$, where*

$$\cot\theta_p = \frac{\sqrt{(p-2)^2 + p^2 c_2^2}}{2\sqrt{p-1}}.$$

**Example 1.** Theorem 1.1 applies to semigroups generated by second order elliptic operators in divergence form with bounded measurable real coefficients:

$$Au = -\operatorname{div} S(x)\nabla u \quad \text{on } \Omega.$$

In this example one takes $\mu = \lambda$, where $\lambda$ is the Lebesgue measure on $\Omega$, so that $L^2_\mu = L^2$ and $H^1_\mu = H^1$ are the usual Lebesgue and Sobolev spaces on $\Omega$. The form $(a, H^1)$ is given by

$$a(u, v) = \int_\Omega \big\langle S(x)\nabla u, \nabla v\big\rangle \,\mathrm{d}\lambda, \quad u, v \in H^1.$$

**Example 2.** Theorem 1.1 applies to semigroups generated by *Ornstein–Uhlenbeck operators* of the form

$$Au = -\Delta u - Bx\nabla u \quad \text{on } \mathbb{R}^N, \tag{4}$$

where $B$ is a real matrix having only eigenvalues with negative real part, or

$$Au = -\operatorname{div} S\nabla u - S^*\nabla\varphi\nabla u \quad \text{on } \Omega, \tag{5}$$

where $\varphi \in C^1(\Omega)$ and $S \in \mathbb{R}^{N\times N}$. Theorem 1.1 applies if $\mu$ is the invariant measure for the Ornstein–Uhlenbeck semigroup, given by

$$\mathrm{d}\mu(x) = \frac{1}{\sqrt{(4\pi)^N \det Q_\infty}} \mathrm{e}^{-\frac{1}{4}\langle Q_\infty^{-1}x,x\rangle}\,\mathrm{d}\lambda(x), \quad Q_\infty := \int_0^\infty \mathrm{e}^{sB}\,\mathrm{e}^{sB^*}\,\mathrm{d}s$$

for (4), and $\mathrm{d}\mu(x) = \mathrm{e}^{-\varphi(x)}\,\mathrm{d}\lambda(x)$ for (5). The form $(a, H^1_\mu)$ is given, respectively, by

$$a(u,v) = -2\int_{\mathbb{R}^N} \langle Q_\infty B^*\nabla u, \nabla v\rangle\,\mathrm{d}\mu, \quad a(u,v) = \int_{\mathbb{R}^N} \langle S\nabla u, \nabla v\rangle\,\mathrm{d}\mu, \quad u,v\in H^1_\mu.$$

**Remark 1.** The angle of analyticity $\theta_p$ from Theorem 1.1 is in general better than the angle of analyticity which one would obtain by the Stein interpolation theorem, see [10]. That angle would be $\theta_2(1-|\frac{2}{p}-1|)$.

**Remark 2.** In the case when the $S(x)$ are symmetric, so that $A_2$ is self-adjoint, Theorem 1.1 has been proved in [1,3,8]. For general symmetric submarkovian semigroups, see [5,6].

**Remark 3.** The choice of our form domain corresponds to Neumann type boundary conditions. The proof of Theorem 1.1 will show that instead of the form domain $H^1_\mu$ one may also choose $H^1_{\mu,0}$ as form domain (the closure of $C_c^\infty(\Omega)$ in $H^1_\mu$), which corresponds to Dirichlet boundary conditions in the case of nondegenerate $m$. In the case of nondegenerate $m$ and Lipschitz regular $\Omega$, and if $\beta \in L^\infty(\partial\Omega)^+$ (w.r.t. the surface measure $\sigma$), one may take also $H^1_\mu = H^1$ as form domain, but change the form to

$$a(u,v) := \int_\Omega \big\langle S(x)\nabla u, \nabla v\big\rangle\,\mathrm{d}\mu + \int_{\partial\Omega} \beta(x)u\bar{v}\,\mathrm{d}\sigma, \quad u,v\in H^1_\mu,$$

which then corresponds to Robin type boundary conditions.

**Remark 4.** Clearly, it can happen that the semigroup $(\mathrm{e}^{-tA_p})_{t\geqslant 0}$ extends analytically to a larger sector than the sector described in Theorem 1.1. This can happen even if $p=2$ and the constant $c_2$ from (1) is optimal; for nonsymmetric but *constant* $S$ one has $c_2>0$ but if $\mu=\lambda$ is the Lebesgue measure on $\Omega$ then the operator $A_2$ is self-adjoint. In this case, it is known that $(\mathrm{e}^{tA_p})_{t\geqslant 0}$ extends analytically to the sector $\Sigma_{\theta_p}$ where $\cot\theta_p = |p-2|/(2\sqrt{p-1})$, which is our $\theta_p$ for $c_2=0$. However, in general the angle $\theta_p$ from Theorem 1.1 is optimal. For symmetric $A$ this follows from [4,11]. For nonsymmetric $A$, it follows from [2, Theorem 2] that if $A$ is the Ornstein–Uhlenbeck operator in (4) on the space $L^p_\mu$ as in Example 2, then for every $p\in(1,\infty)$ the angle of analyticity $\theta_p$ from Theorem 1.1 is optimal. More precisely: whenever $(\mathrm{e}^{-tA_p})_{t\geqslant 0}$ extends to an analytic semigroup on a sector $\Sigma_\theta$ (the extended semigroup need a priori not be a contraction semigroup), then $\theta \leqslant \theta_p$.

**Proof of Theorem 1.1.** Fix $p\in(2,\infty)$. By the Lumer–Phillips theorem, [7], the semigroup $(\mathrm{e}^{-tA_p})_{t\geqslant 0}$ on $L^p_\mu$ extends to an analytic semigroup of contractions on the sector $\Sigma_{\theta_p}$ if and only if $-\mathrm{e}^{i\varphi}A_p$ is dissipative for every $\varphi\in(-\theta_p,\theta_p)$, i.e. if and only if for every $u\in D(A_p)$

$$\left|\mathrm{Im}\int_\Omega A_p u u^*\,\mathrm{d}\mu\right| \leqslant \cot\theta_p\,\mathrm{Re}\int_\Omega A_p u u^*\,\mathrm{d}\mu, \quad \text{where } u^* := \bar{u}|u|^{p-2}. \tag{6}$$

Note that for every $u\in D := D(A_2)\cap D(A_p)\cap L^\infty$ one has $u\in H^1_\mu\cap L^\infty$ and therefore also $u^*\in H^1_\mu\cap L^\infty$. Hence, for every $u\in D$ one has

$$\int_\Omega A_p u u^*\,\mathrm{d}\mu = \int_\Omega A_2 u u^*\,\mathrm{d}\mu = a(u,u^*) = \int_\Omega S(x)\nabla u\nabla u^*\,\mathrm{d}\mu;$$

here, $\xi\eta = \sum_{i=1}^N \xi_i\eta_i$ for $\xi, \eta \in \mathbb{C}^N$. Since $D$ is a core for $A_p$ (note that $D$ is dense in $L^p_\mu$ and invariant under the semigroup), inequality (6) holds for every $u \in D(A_p)$ if and only if for every $u \in D$ one has

$$\left|\operatorname{Im}\int_\Omega S(x)\nabla u\nabla u^*\,\mathrm{d}\mu\right| \leqslant \cot\theta_p \operatorname{Re}\int_\Omega S(x)\nabla u\nabla u^*\,\mathrm{d}\mu. \tag{7}$$

Fix $x \in \Omega$ and $u \in D$. Set $S := S(x)$, and let $S_1 := (S + S^*)/2$ and $S_2 := (S - S^*)/2$ be the symmetric and the antisymmetric part of $S$, respectively. Write $u = v + iw$, where $v$ and $w$ are real-valued. If $u^*$ is defined as in (6), then $\nabla u^* = \nabla\bar{u}|u|^{p-2} + (p-2)\bar{u}(v\nabla v + w\nabla w)|u|^{p-4}$. Writing $|u|^{p-2} = |u|^{p-4}(v^2 + w^2)$, we thus obtain

$$\begin{aligned} S\nabla u\nabla u^* &= |u|^{p-4}\big(v^2+w^2\big)\big(S(\nabla v + \mathrm{i}\nabla w)(\nabla v - \mathrm{i}\nabla w)\big) - |u|^{p-4}(v - \mathrm{i}w)\big(S(\nabla v + \mathrm{i}\nabla w)(v\nabla v + w\nabla w)\big) \\ &\quad + (p-1)|u|^{p-4}(v - \mathrm{i}w)\big(S(\nabla v + \mathrm{i}\nabla w)(v\nabla v + w\nabla w)\big). \end{aligned}$$

By simplifying, we obtain

$$\begin{aligned} S\nabla u\nabla u^* &= |u|^{p-4}\big[w^2 S\nabla v\nabla v + v^2 S\nabla w\nabla w + \mathrm{i}(w^2 S\nabla w\nabla v - v^2 S\nabla v\nabla w) \\ &\quad - vwS\nabla v\nabla w - vwS\nabla w\nabla v + \mathrm{i}(vwS\nabla v\nabla v - vwS\nabla w\nabla w) \\ &\quad + (p-1)\big(v^2 S\nabla v\nabla v + vwS\nabla v\nabla w\big) + \mathrm{i}(p-1)\big(v^2 S\nabla w\nabla v + vwS\nabla w\nabla w\big) \\ &\quad + (p-1)\big(vwS\nabla w\nabla v + w^2 S\nabla w\nabla w\big) - \mathrm{i}(p-1)\big(vwS\nabla v\nabla v + w^2 S\nabla v\nabla w\big)\big] \\ &= |u|^{p-4}\big[S(v\nabla w - w\nabla v)(v\nabla w - w\nabla v) + (p-1)S(v\nabla v + w\nabla w)(v\nabla v + w\nabla w) \\ &\quad + \mathrm{i}(p-1)S(v\nabla w - w\nabla v)(v\nabla v + w\nabla w) - \mathrm{i}S(v\nabla v + w\nabla w)(v\nabla w - w\nabla v)\big]. \end{aligned}$$

Observe that $S\xi\eta = S^*\eta\xi$ for every $\xi, \eta \in \mathbb{R}^N$, and that $\operatorname{Re}(\bar{u}\nabla u) = v\nabla v + w\nabla w$, $\operatorname{Im}(\bar{u}\nabla u) = v\nabla w - w\nabla v$. Hence

$$\begin{aligned} S\nabla u\nabla u^* &= |u|^{p-4}\big[\big\langle S\operatorname{Im}(\bar{u}\nabla u), \operatorname{Im}(\bar{u}\nabla u)\big\rangle + (p-1)\big\langle S\operatorname{Re}(\bar{u}\nabla u), \operatorname{Re}(\bar{u}\nabla u)\big\rangle \\ &\quad + \mathrm{i}\big\langle\big((p-1)S - S^*\big)\operatorname{Im}(\bar{u}\nabla u), \operatorname{Re}(\bar{u}\nabla u)\big\rangle\big]. \end{aligned}$$

Since $\langle S\xi, \xi\rangle = \langle S_1\xi, \xi\rangle$ for every $\xi \in \mathbb{R}^N$, and $(p-1)S - S^* = (p-2)S_1 - pS_2$, we finally obtain

$$\begin{aligned} S\nabla u\nabla u^* &= |u|^{p-4}\big[\big\langle S_1\operatorname{Im}(\bar{u}\nabla u), \operatorname{Im}(\bar{u}\nabla u)\big\rangle + (p-1)\big\langle S_1\operatorname{Re}(\bar{u}\nabla u), \operatorname{Re}(\bar{u}\nabla u)\big\rangle \\ &\quad + \mathrm{i}\big\langle\big((p-2)S_1 - pS_2\big)\operatorname{Im}(\bar{u}\nabla u), \operatorname{Re}(\bar{u}\nabla u)\big\rangle\big]. \end{aligned}$$

Since $S_1$ is elliptic, there exist $S_1^{1/2}$ and $S_1^{-1/2}$. By the Cauchy–Schwarz inequality

$$\begin{aligned} |\operatorname{Im} S\nabla u\nabla u^*| &= |u|^{p-4}\big|\big\langle\big((p-2)I - pS_1^{-1/2}S_2S_1^{-1/2}\big)\big(S_1^{1/2}\operatorname{Im}(\bar{u}\nabla u)\big), \big(S_1^{1/2}\operatorname{Re}(\bar{u}\nabla u)\big)\big\rangle\big| \\ &\leqslant |u|^{p-4}\big\|(p-2)I - pS_1^{-1/2}S_2S_1^{-s\frac{1}{2}}\big\|\big\|S_1^{1/2}\operatorname{Im}(\bar{u}\nabla u)\big\|\big\|S_1^{1/2}\operatorname{Re}(\bar{u}\nabla u)\big\|. \end{aligned}$$

On the other hand,

$$\operatorname{Re} S\nabla u\nabla u^* = |u|^{p-4}\big(\big\|S_1^{1/2}\operatorname{Im}(\bar{u}\nabla u)\big\|^2 + (p-1)\big\|S_1^{1/2}\operatorname{Re}(\bar{u}\nabla u)\big\|^2\big).$$

Since the matrix $S_1^{-1/2}S_2S_1^{-1/2}$ is skew-adjoint, the norm of the normal matrix $(p-2)I - pS_1^{-1/2}S_2S_1^{-1/2}$ is equal to its spectral radius. The latter can be easily computed by using Pythagoras' theorem and one obtains

$$\big\|(p-2)I - pS_1^{-1/2}S_2S_1^{-1/2}\big\| = \sqrt{(p-2)^2 + p^2\big\|S_1^{-1/2}S_2S_1^{-1/2}\big\|^2}.$$

By assumption (1), $\|S_1^{-1/2}S_2S_1^{-1/2}\| \leqslant c_2$, so that

$$\big\|(p-2)I - pS_1^{-1/2}S_2S_1^{-1/2}\big\| \leqslant \sqrt{(p-2)^2 + p^2c_2^2} =: \kappa.$$

It is easy to verify that for

$$\gamma := \frac{\sqrt{(p-2)^2 + p^2c_2^2}}{2\sqrt{p-1}}$$

one has $\kappa ab \leqslant \gamma(a^2 + (p-1)b^2)$ for every $a, b \geqslant 0$. Hence, we have proved that for every $x \in \Omega$ and every $u \in D$, $|\mathrm{Im}\, S(x)\nabla u \nabla u^*| \leqslant \gamma\, \mathrm{Re}\, S(x)\nabla u \nabla u^*$. Integrating this inequality over $\Omega$, we obtain (7).

Now let $p \in (1,2)$, let $p'$ be the conjugate exponent and $A_p^*$ be the adjoint operator. Notice that $A_2^*$ is obtained as $A_2$, starting from $S^*$ instead of $S$, and that the constant $c_2$ in (1) is the same for $S$ and $S^*$. Moreover, $A_p^*$ is obtained by extrapolation from $A_2^*$ with the same procedure as $A_{p'}$, hence we may apply the first part of the proof, obtaining for $A_p^*$ an analyticity angle $\theta_{p'} = \theta_p$. By duality, the angles of analyticity for $A_p$ and $A_p^*$ coincide, and the claim is proved. □

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