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Partial Differential Equations

Dynamics of multiple degree Ginzburg–Landau vortices

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Abstract

For the two-dimensional complex parabolic Ginzburg–Landau equation we prove that, asymptotically, vortices evolve according to a simple ordinary differential equation, which is a gradient flow of the Kirchhoff–Onsager functional. This convergence holds except for a finite number of times, corresponding to vortex collisions and splittings, which we describe carefully. The only assumption is a natural energy bound on the initial data. ***To cite this article: F. Bethuel et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Dynamique des tourbillons de vorticité de degré multiple pour l'équation de Ginzburg–Landau. Nous montrons, pour l'équation de Ginzburg–Landau parabolique complexe en dimension deux, qu'asymptotiquement les tourbillons se déplacent suivant un flot gradient pour la fonctionnelle de Kirchhoff–Onsager. Cette convergence a lieu en dehors d'un nombre fini d'instants qui correspondent aux éclatements et aux collisions des tourbillons, que nous décrivons en détail. Notre unique hypothèse sur les données initiales est une borne d'énergie. ***Pour citer cet article : F. Bethuel et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Les résultats présentés dans cette Note parachèvent notre étude entamée dans [1,2] de l'équation de Ginzburg–Landau parabolique $(\mathrm{PGL})_\varepsilon$ en dimension deux d'espace. L'objectif de nos travaux est de décrire le comportement asymptotique des solutions lorsque le paramètre ε tend vers 0, sous l'unique hypothèse (H_0) portant sur l'énergie des données initiales. Il a été constaté qu'une échelle de temps accélérée par un facteur $|\log\varepsilon|$ est particulièrement appropriée pour décrire la dynamique de la vorticité, et que dans cette échelle de temps la convergence (1) a lieu. Cette convergence permet de donner un sens précis à la notion de tourbillons : ceux-ci sont identifiés avec les points $a_i(s)$ et leurs charges sont données par les entiers relatifs $d_i(s)$. Nos résultats fournissent une description détaillée de leurs trajectoires. Nous montrons d'abord :

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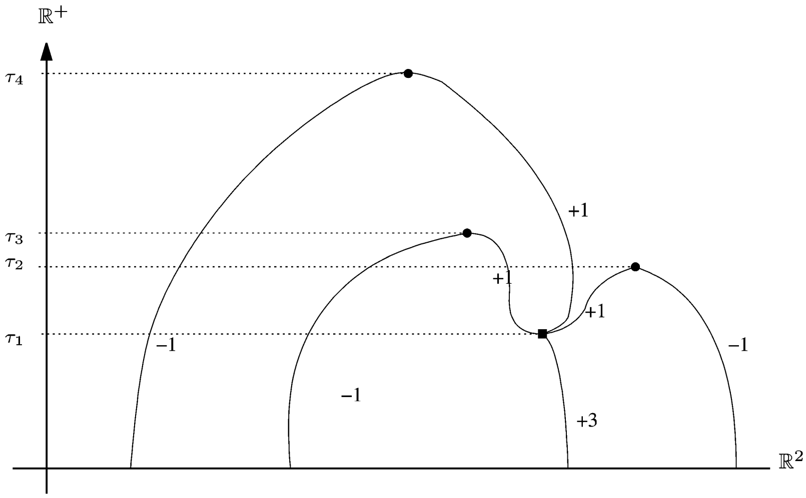

Fig. 1. An example of trajectory set.

Fig. 1. Un exemple de trajectoire.

Théorème 1. *Il existe un nombre fini d'instants* $0 = \tau_0 < \tau_1 < \cdots < \tau_q < \tau_{q+1} = +\infty$ *tels que*

(i) *Le nombre de tourbillons* $\ell(s) \equiv \ell_k$ *est constant sur chaque intervalle* (τ_k, τ_{k+1}), *pour* $k = 0, \ldots, q$.
(ii) *La restriction de l'ensemble* $\Sigma_{\mathfrak{v}}$ *défini par* (2) *à* $\mathbb{R}^2 \times (\tau_k, \tau_{k+1})$ *est une union finie de* ℓ_k *courbes régulières disjointes. Plus précisément, quitte à renuméroter les points* $a_1(s), \ldots, a_{\ell_k}(s)$, *leurs degrés* $d_i(s) = d_i$ *sont constants sur* (τ_k, τ_{k+1}), *et leurs trajectoires sont données par le système d'équations différentielles ordinaires* (3), *où* W *désigne la fonctionnelle de Kirchhoff–Onsager définie par* (4).

L'origine de la fonction $\vec{c}$ présente dans (1) est à chercher dans les oscillations de très basse fréquence de la donnée initiale. Lorsque cette fonction est identiquement nulle, le système différentiel (3) s'apparente à un flot gradient pour la fonctionnelle de Kirchhoff–Onsager, pour la métrique pondérée par les degrés d_i. Cette propriété s'avère cruciale dans notre démonstration de la majoration (18). Lorsque $\vec{c}$ n'est pas nulle, elle agit dans (3) comme un terme de dérive pour les tourbillons.

Les instants $\tau_1, \ldots, \tau_q$ ont été identifiés dans [2] comme les uniques instants de dissipation de l'énergie, comme il est indiqué dans (5) et (6). Notre deuxième résultat identifie les points où la dissipation se concentre avec les points de branchement de l'ensemble des trajectoires des tourbillons (voir Fig. 1).

Théorème 2. *Un point* $(a_i(\tau_k), \tau_k)$ *est un point de branchement si et seulement si il est un point de dissipation, auquel cas les relations* (8) *ont lieu.*

La preuve des Théorèmes 1 et 2 repose de manière cruciale sur une identité algébrique (15) concernant la fonctionnelle de Kirchhoff–Onsager. Cette identité, dont nous ignorons si elle était connue par ailleurs, permet de retrouver une relation classique due à Kirchhoff [5], qui affirme que pour un point critique de W, la somme des carrés des charges est égale au carré de la somme des charges. Grâce à (15), nous montrons que pour ces mêmes points critiques les barycentres (13) et (10) pondérés respectivement par les charges et par les charges au carré coïncident.

1. Introduction

The results presented in this note complete a series of works [1,2] devoted to the study of the two-dimensional complex-valued parabolic Ginzburg–Landau equation

$$\frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon \big(1 - |u_\varepsilon|^2\big) \quad \text{on } \mathbb{R}^2 \times \mathbb{R}^+. \tag{PGL}_\varepsilon$$

Our focus is put on the description of the asymptotic behavior of sequences of solutions as $\varepsilon \to 0$, under the only assumption that the initial datum $u_\varepsilon^0(\cdot) \equiv u_\varepsilon(\cdot, 0)$ verifies

$$E_\varepsilon\big(u_\varepsilon^0\big) = \int_{\mathbb{R}^2} e_\varepsilon\big(u_\varepsilon^0\big) = \int_{\mathbb{R}^2} \left[\frac{|\nabla u_\varepsilon^0|^2}{2} + \frac{(1-|u_\varepsilon^0|^2)^2}{4\varepsilon^2}\right] \leqslant M_0|\log\varepsilon|, \tag{H$_0$}$$

where $M_0 > 0$ is some fixed constant.

This problem has received a lot of attention in the last decade. It has been recognized that the dynamics is of particular interest accelerating time by a factor $|\log\varepsilon|$, that is considering the functions

$$\mathfrak{u}_\varepsilon(z, s) = u_\varepsilon\big(z, s|\log\varepsilon|\big), \quad z = x + \mathrm{i}y \equiv (x, y) \in \mathbb{R}^2.$$

In this situation, it has been proved in [1,2] (following earlier results in [6,4]) that if $(u_\varepsilon)_{\varepsilon>0}$ verifies (PGL)$_\varepsilon$ and (H$_0$), then for a subsequence $\varepsilon_n \to 0$ we have

$$\mathfrak{u}_{\varepsilon_n}(z, s) \to \mathfrak{u}_*(z, s) = \prod_{i=1}^{l(s)} \left(\frac{z - a_i(s)}{|z - a_i(s)|}\right)^{d_i(s)} \exp\big(\mathrm{i}\big[\langle\vec{c}(s), z\rangle + b(s)\big]\big), \tag{1}$$

where, for $i = 1, \ldots, \ell(s)$, $a_i(s) \in \mathbb{R}^2$, $d_i(s) \in \mathbb{Z}$, $b(s) \in [0, 2\pi)$ and $\vec{c} : \mathbb{R}^+ \to \mathbb{R}^2$ is a Lipschitz function. The convergence in (1) is uniform on every compact subset of $\mathbb{R}^2 \times \mathbb{R}^+ \setminus \Sigma_{\mathfrak{v}}$, where

$$\Sigma_{\mathfrak{v}} = \bigcup_{s>0} \bigcup_{i=1}^{l(s)} \big\{\big(a_i(s), s\big)\big\}. \tag{2}$$

Moreover, the trajectory set $\Sigma_{\mathfrak{v}}$ is a closed, 1-dimensional rectifiable subset of $\mathbb{R}^2$. Notice that the limiting map $\mathfrak{u}_*(\cdot, s)$ has modulus 1, hence with values in the circle S^1, but is singular at the points $a_i(s)$ when $d_i(s) \neq 0$. In this case, it also has diverging local Dirichlet energy. The points $a_i(s)$ are called the vortices at time s, and the integers $d_i(s)$ their degrees: they correspond to the winding numbers of the limiting map $\mathfrak{u}_*(\cdot, s)$ around the vortices $a_i(s)$. The set $\Sigma_{\mathfrak{v}}$ describes the evolution in time of the set of vortices, and therefore we refer to it as the trajectory set. The results of this note provide a complete description of the trajectory set $\Sigma_{\mathfrak{v}}$. We first have

Theorem 1. *There exists a finite number of times* $0 = \tau_0 < \tau_1 < \cdots < \tau_q < \tau_{q+1} = +\infty$ *such that*

(i) *The number of vortices* $\ell(s) \equiv \ell_k$ *is constant on each interval* (τ_k, τ_{k+1})*, for* $k = 0, \ldots, q$.
(ii) *The restriction of* $\Sigma_{\mathfrak{v}}$ *to* $\mathbb{R}^2 \times (\tau_k, \tau_{k+1})$ *is a disjoint union of* ℓ_k *smooth one-dimensional graphs.*

More precisely, relabeling possibly the points $a_1(s), \ldots, a_{\ell_k}(s)$*, their degrees* $d_i(s) = d_i$ *are constant in* (τ_k, τ_{k+1})*, and their trajectories are given by the system of ordinary differential equations*

$$d_i^2 \frac{\mathrm{d}a_i}{\mathrm{d}s}(s) = -\nabla_{a_i} W(a_1, \ldots, a_{\ell_k}) + d_i c(s)^\perp, \quad i = 1, \ldots, \ell_k, \tag{3}$$

where W is the Kirchhoff–Onsager function defined as

$$W(a_1, \ldots, a_{\ell_k}) = -2 \sum_{i\neq j=1}^{\ell_k} d_i d_j \log|a_i - a_j|. \tag{4}$$

The times $\tau_1, \ldots, \tau_q$ were already identified in [2] as the only times of dissipation in an appropriate asymptotic sense. More precisely, it was proved in [2] Theorem 4 and Corollary 3.1 that for $s \notin \{\tau_0, \ldots, \tau_q\}$,

$$\mathfrak{v}_{\varepsilon_n}^s(x) \equiv \frac{e_{\varepsilon_n}(\mathfrak{u}_{\varepsilon_n}(x, s))}{|\log\varepsilon_n|} \,\mathrm{d}x \rightharpoonup \mathfrak{v}_*^s = \pi \sum_{i=1}^{\ell(s)} d_i^2(s) \delta_{a_i(s)} \tag{5}$$

in the sense of measures on $\mathbb{R}^2$, and

$$|\partial_t u_{\varepsilon_n}|^2 \,\mathrm{d}x\,\mathrm{d}s \rightharpoonup \omega_* = \pi \sum_{k=0}^{q} \sum_{i=1}^{\ell(\tau_k)} \beta_i^k \delta_{(a_i(\tau_k), \tau_k)}, \tag{6}$$

in the sense of measures on $\mathbb{R}^2 \times \mathbb{R}^+$, where $\beta_i^k \in \mathbb{N}$. Since the total energy $\pi \sum d_i^2(s)$ is quantized and non increasing (see [2]), it is also piecewise constant. The times $\tau_1, \ldots, \tau_q$ correspond therefore to the times of energy loss, where dissipation concentrates. The points $(a_i(\tau_k), \tau_k)$ for which $\beta_i^k \neq 0$ are called the *dissipation points*.

Theorem 1 completely describes the trajectories inside the intervals (τ_k, τ_{k+1}) (see also Fig. 1). The next step is to understand the behavior of the trajectories across the dissipation times. Since $\Sigma_{\mathfrak{v}}$ is closed, the points $(a_i(\tau_k), \tau_k)$ are the only possible endpoints of the trajectories in (τ_{k-1}, τ_k) and (τ_k, τ_{k+1}). For a given point $(a_i(\tau_k), \tau_k)$, let $\mathcal{C}_1^-, \ldots, \mathcal{C}_{l_i^-}^-$ and $\mathcal{C}_1^+, \ldots, \mathcal{C}_{l_i^+}^+$ denote the vortices trajectories respectively in $\mathbb{R}^2 \times (\tau_{k-1}, \tau_k)$ and $\mathbb{R}^2 \times (\tau_k, \tau_{k+1})$ for which $(a_i(\tau_k), \tau_k)$ is an endpoint. Accordingly, let $d_1^-, \ldots, d_{l_i^-}^-$ and $d_1^+, \ldots, d_{l_i^+}^+$ be the degrees of the corresponding vortices. It follows from (5), (6) and the fact that $(\mathrm{PGL})_\varepsilon$ is a gradient flow that

$$\beta_i^k = \sum_{j=1}^{l_i^-} \left(d_j^-\right)^2 - \sum_{j=1}^{l_i^+} \left(d_j^+\right)^2. \tag{7}$$

We say that a point $(a_i(\tau_k), \tau_k)$ is a *regular point* of the trajectory set if $\Sigma_{\mathfrak{v}}$ is a Lipschitz graph over s in the neighborhood of $(a_i(\tau_k), \tau_k)$, or equivalently if $l_i^- = l_i^+ = 1$. If not, we say that $(a_i(\tau_k), \tau_k)$ is a *branching point*.

Theorem 2. *A point $(a_i(\tau_k), \tau_k)$ is a branching point if and only if it is a dissipation point. Moreover*

$$\sum_{j=1}^{l_i^-} d_j^- = d_i(\tau_k) = \sum_{j=1}^{l_i^+} d_j^+ \quad \text{and} \quad \sum_{j=1}^{l_i^-} \left(d_j^-\right)^2 \geqslant d_i^2(\tau_k) \geqslant \sum_{j=1}^{l_i^+} \left(d_j^+\right)^2 \tag{8}$$

where the first (resp. second) inequality is strict whenever $l_i^- \geqslant 2$ (resp. $l_i^+ \geqslant 2$).

Part of the statement in the previous theorems have already been known in the case $|d_i| = 1$ [6,4,7,1,2,8]. An important novelty here is that there are no restriction on the d_i's. Whether multiple degrees may be really observed as limits of solutions to $(\mathrm{PGL})_\varepsilon$ is positively answered in [3].

A natural question is to know if the positions and degrees of the vortices at some time s_0 completely determine their positions and degrees at future times. Whereas collisions are determined by singularities in the ordinary differential equation (3), splittings are not, and clearly Theorem 1 does not settle the question of the occurrence of such splittings. More precisely, the dissipation times $\tau_1, \ldots, \tau_q$ are not inferred in a constructive way. As a matter of fact, there is no hope to determine the complete future trajectories by knowing the positions and the degrees at some fixed time: an important part of the relevant information is lost in the limiting procedure.

The detailed proofs are presented in [3].

2. The new ingredient

The main point is statement (i) in Theorem 1, i.e. to prove that the number of vortices is locally constant, except for dissipation times. Statement (ii) and Theorem 2 then essentially follow from the analysis in [1,2]. The difficulty which remained unsolved, and which is equivalent to statement (i), is to show that except for dissipation times multiple degree vortices do not split nor collide. This amounts to show that for $s_0 \notin \{\tau_1, \ldots, \tau_q\}$ and $(a_i(s_0), s_0) \in \Sigma_{\mathfrak{v}}$, for s close to s_0 and for some neighborhood B_i of $a_i(s_0)$, B_i contains only a single vortex.

In order to analyze the size and the spreading of the possible cluster of vortices emanating from or colliding at $(a_i(s_0), s_0)$, we consider the variance

$$f_i(s) = \frac{\sum_{a_j(s) \in B_i} d_j^2(s) |a_j(s) - \hat{a}_i(s)|^2}{\sum_{a_j(s) \in B_i} d_j^2(s)}, \tag{9}$$

where $\hat{a}_i(s)$ denotes the barycenter of the cluster of vortices in B_i, with weights given by the energy densities $d_j^2(s)$, namely

$$\hat{a}_i(s) = \frac{\sum_{a_j(s) \in B_i} d_j^2(s) a_j(s)}{\sum_{a_j(s) \in B_i} d_j^2(s)}. \tag{10}$$

Clearly $f_i(s_0)=0$, and our goal is to prove that $f_i(s)$ vanishes identically in a neighborhood of s_0, by mean of a Gronwall type inequality. In view of (5) and the fact that s_0 is not a dissipation time, one obtains that for s close to s_0 the Kirchhoff law

$$\sum_{a_j(s)\in B_i} d_j^2(s)=\Bigl(\sum_{a_j(s)\in B_i} d_j(s)\Bigr)^2=d_i^2(s_0) \tag{11}$$

is satisfied. On the other hand, convergence (1), quantization (5) and (6) and the evolution law for the energy densities in $(\mathrm{PGL})_\varepsilon$ allow to derive the formula

$$f_i'(s)=\frac{4}{d_i^2(s_0)}\sum_{\substack{a_k(s)\notin B_i\\ a_j(s)\in B_i}} d_k(s)d_j(s)\mathcal{R}e\left(\frac{a_j(s)-\hat{a}_i(s)}{a_k(s)-a_j(s)}\right)+\frac{2}{d_i(s_0)}\bigl\langle\check{a}_i(s)-\hat{a}_i(s),\vec{c}(s)^\perp\bigr\rangle, \tag{12}$$

where $\check{a}_i(s)$ denotes a second type of barycenter, namely

$$\check{a}_i(s)=\frac{\sum_{a_j(s)\in B_i} d_j(s)a_j(s)}{\sum_{a_j(s)\in B_i} d_j(s)}. \tag{13}$$

One deduces from (12) that f_i is a Lipschitz function near s_0 and verifies

$$\bigl|f_i'(s)\bigr|\leqslant C(M_0,s_0)\bigl(\bigl|\hat{a}_i(s)-\check{a}_i(s)\bigr|+f_i(s)\bigr). \tag{14}$$

In order to integrate the differential inequality (14), we need some control on the term $|\check{a}_i(s)-\hat{a}_i(s)|$. For arbitrary configurations of points and degrees, there is no reason that $\hat{a}$ and $\check{a}$ should be close. However, for simple examples of critical points of W we noticed that they are equal. This observation led us to the following identity for W.

Lemma. *Consider ℓ points $z_1,\dots,z_\ell\in\mathbb{C}$, and ℓ real numbers $d_1,\dots,d_l$ whose sum is non zero. Then the following identity holds*:

$$\frac{\sum d_jz_j}{\sum d_j}=\frac{\sum d_j^2z_j}{(\sum d_j)^2}+\frac{\sum\nabla_{z_j}W(z_1,\dots,z_\ell)z_j^2}{2(\sum d_j)^2}, \tag{15}$$

where the sums are meant for j ranging from 1 *to* ℓ.

Specifying formula (15) with $z_j=a_j(s)-\hat{a}_i(s)$, and in view of (11), we obtain the inequality

$$\bigl|\check{a}_i(s)-\hat{a}_i(s)\bigr|\leqslant C(M_0)\bigl|\nabla W\bigl(\{a_j(s)\}_{j\in I(s)}\bigr)\bigr|f_i(s), \tag{16}$$

where $I(s)=\{j\in\{1,\dots,\ell(s)\},\ a_j(s)\in B_i\}$. Combining (14) and (16), we finally derive

$$\bigl|f_i'(s)\bigr|\leqslant C(M_0,s_0)\bigl(1+\bigl|\nabla W\bigl(\{a_j(s)\}_{j\in I(s)}\bigr)\bigr|\bigr)f_i(s). \tag{17}$$

Since $f_i(s_0)=0$, Gronwall's lemma would then allow to conclude that $f_i(s)\equiv0$ on a neighborhood I of s_0 provided that

$$\int_I\bigl|\nabla W\bigl(\{a_j(s)\}_{j\in I(s)}\bigr)\bigr|\,\mathrm{d}s<+\infty. \tag{18}$$

As a matter of fact, we even prove that $|\nabla W|\in L^2(I)$, using the gradient-flow type properties of the ode (3). This last statement may seem rather odd at first reading, since the ode (3) is precisely what we wish to show. Our actual argument is by induction on $d_i(s_0)$. Indeed, when $|d_i(s_0)|=2$, the splitting may only create ±1 vortices, for which we already established (3) in [2]. Similarly, if $|d_i(s_0)|=k$, the splitting may only involve vortices of degree at most $k-1$ in absolute value, which are handled by the inductive argument.

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