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Calculus of Variations

On Lipschitz regularity of minimizers of a calculus of variations problem with non locally bounded Lagrangians $\stackrel{\circ}{\approx}$

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Abstract

We prove that the optimal solutions of a calculus of variations problem are Lipschitz continuous. The result is obtained without assuming that the domain of the Lagrangian is the whole space as usually stated in the literature. So, the contribution of this Note is in giving a new sufficient condition for the nonexistence of a Lavrentiev phenomenon. *To cite this article: M. Quincampoix, N. Zlateva, C. R. Acad. Sci. Paris, Ser. I 343 (2006).*

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Résumé

Sur la régularité lipschitzienne des solutions d'un problème de calcul des variations avec lagrangiens non localement bornés. Dans cette Note, nous prouvons que les solutions optimales d'un problème de calcul des variations sont lipschitziennes. Ce résultat est obtenu sans supposer, comme souvent dans la littérature, que le lagrangien est défini sur tout l'espace. Cet article donne donc une nouvelle condition suffisante pour l'absence de phénomène de Lavrentieff. *Pour citer cet article : M. Quincampoix, N. Zlateva, C. R. Acad. Sci. Paris, Ser. I 343 (2006).*

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Nous considérons le problème de calcul des variations suivant

$$V := \min\left\{\int_{0}^{1} L(y(t), y'(t)) dt \mid y \in W^{1,1}([0, T]; \mathbb{R}^{n}), y(0) = 0\right\},$$
(1)

où L est un lagrangien. L'existence de solutions optimales absolument continues de (1) est bien connue depuis Tonelli [12], mais parfois, elles peuvent ne pas être lipschitziennes, c'est ce que l'on appelle le phénomène de Lavrentieff [3]. L'existence de solutions lipschitziennes est importante à la fois pour obtenir des conditions d'Euler mais aussi

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pour caractériser la valeur par des équations d'Hamilton Jacobi Bellman [6,10]. Dans la littérature récente il existe un grand nombre de travaux consacrés à l'absence du phénomène de Lavrentieff, mais de façon surprenante, peu d'entre eux étudient le cas où le Lagrangien n'est pas défini sur tout l'espace. En effet, la motivation principale pour étudier ce cas provient du contrôle (modélisé par une inclusion différentielle) à dynamique non nécessairement bornée : le problème de contrôle optimal

$$\min\left\{\int_{0}^{1} l(y(t), y'(t)) dt \mid y \in W^{1,1}([0, T]; \mathbb{R}^{n}), y(0) = 0, y'(t) \in \Phi(y(t)), t \in [0, T]\right\},\$$

se ramène au problème (1) à condition de poser L(x, v) = l(x, v) si $v \in \Phi(x)$ et $L(x, v) = +\infty$ sinon. Le cas où l'ensemble des vitesses possibles $\Phi(y(t))$ est ni compact, ni \mathbb{R}^n , est donc le cas intéressant pour notre étude. Dans cet article, en améliorant un résultat de [7], nous proposons une condition suffisante pour l'absence de phénomène de Lavrentieff qui est formulée dans le théorème suivant :

Théorème. Soit $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$. Supposons que

(a) L vérifie

 $L(x, v) \ge \Theta(v), \quad \forall (x, v) \in \mathbb{R}^n \times \mathbb{R}^n,$

- $o\dot{u} \Theta : \mathbb{R}^n \to \mathbb{R}_+$ est une fonction mesurable vérifiant $\lim_{\|v\|\to\infty} \Theta(v) \|v\|^{-1} = +\infty$;
- (b) L est bornée sur les ensembles bornés inclus dans son domaine;
- (c) La correspondance-domaine $x \rightrightarrows D(x) := \text{dom } L(x, \cdot)$ est à valeurs fermées, convexes, non vides;
- (d) La correspondance-épigraphique $x \rightrightarrows F(x) := epi L(x, \cdot)$ est k-Lipschitzienne à valeurs fermées.

Alors toute solution $y(\cdot)$ de (1) est lipschitzienne.

1. Introduction and preliminaries

We consider the following calculus of variations problem

$$V := \min\left\{\int_{0}^{1} L(y(t), y'(t)) dt \mid y \in W^{1,1}([0, T]; \mathbb{R}^{n}), y(0) = 0\right\},$$
(1)

where $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ is called a Lagrangian. The existence of absolutely continuous minimizers is wellstudied since pioneering works of Tonelli (cf. [12,5]). However, sometimes, the minimizers fails to be Lipschitz: this is the so-called Lavrentiev phenomenon [3]. The existence of Lipschitz minimizers is important mainly for two reasons: first, it enables us to derive necessary optimality conditions [5], and second, it also allows to obtain characterization of the value function associated with (1) through Hamilton Jacobi Bellman equations (cf. [6,10] for recent works in this field). In the literature there exists a large number of sufficient conditions for nonexistence of Lavrentiev phenomenon, but surprisingly, up to the knowledge of the authors of the present note, all these conditions require of the second Lagrangian variable domain to be the whole space.¹ To go further that case, our motivation comes from the control theory: the following control Bolza problem with non necessarily bounded dynamic

$$\min\left\{\int_{0}^{T} l(y(t), y'(t)) dt \mid y \in W^{1,1}([0, T]; \mathbb{R}^{n}), \ y(0) = 0, \ y'(t) \in \Phi(y(t)), \ t \in [0, T]\right\},\$$

reduces to (1) if we set L(x, v) = l(x, v) for $v \in \Phi(x)$ and $L(x, v) = +\infty$, otherwise. The interesting case to treat is those where the set of admissible velocities $\Phi(y(t))$ is neither compact nor \mathbb{R}^n . For example,

¹ In [1], Theorem 4.2, Lipschitz regularity is obtained when the minimizers are such that $y'(t) \in \text{int dom } L(y(t), \cdot)$. Here we do not need such an assumption.

$$\min\left\{\int_{0}^{1} \left\|x'(t)\right\|^{2} \mathrm{d}t \ \left|\ x(0) = 0, \ x'(t) \in x(t) + C, \ t \in [0,1], \ x'(t) \in L^{1}([0,1], \mathbb{R}^{n})\right\}\right\},\$$

where $C \subset \mathbb{R}^n$ is a closed, convex and unbounded set with empty interior, neither the results in [1], nor those in [7] does answer the question whether the minimizers are Lipschitz, while Theorem 2.1 below do.

The aim of the present Note is to provide in Theorem 2.1 a new sufficient condition for nonexistence of the Lavrentiev phenomenon for non everywhere finite Lagrangians, extending in this way the results known for Lagrangians finite-valued on the whole space. The most of these results are discussed in the recent paper [7] and Theorem 2.1 therein can be considered as their good generalization.

The existence of a solution to (1) is a consequence of the direct method of the calculus of variations [12,5,9,8]. Among results for nonexistence of Lavrentiev phenomenon we quote one of the more recent:

Proposition 1.1 ([7], Theorem 2.1). Let $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a Borel function which satisfies

$$L(x, v) \ge \Theta(v), \quad \forall (x, v) \in \mathbb{R}^n \times \mathbb{R}^n, \text{ where the function } \Theta : \mathbb{R}^n \to \mathbb{R}_+ \text{ is such that}$$
 (2)

$$\lim_{\|v\|\to\infty} \Theta(v) \|v\|^{-1} = +\infty, \quad and, \text{ moreover}$$
(3)

$$\forall x_0 \in \mathbb{R}^n, \ \exists M > 0, \ \exists r > 0, \ such \ that \ L(x, v) \leqslant M, \ \forall (x, v) \in B[x_0, r] \times B[0, r], \tag{4}$$

where $B[x_0, r]$ stands for the closed ball with center x_0 and radius r.

Then, if $V < +\infty$ and $y \in W^{1,1}([0, T]; \mathbb{R}^n)$ is a solution of (1), then y is necessarily Lipschitz continuous.

When $v \mapsto L(x, \cdot)$ is convex it was proved in [1] that every minimizer of (1) is Lipschitz. When L is continuous, the same result is obtained in [4] under a slightly weaker growth condition. As a further generalization, in [7] the authors show that the convexity hypothesis can be removed, and no continuity assumption is needed. Note that in [7], L is supposed to be everywhere finite-valued. Our intention is to extend such results for Lagrangian which possibly takes infinite values.

Let us recall some basic notions of set-valued analysis. Let \mathbb{R}^n be equipped with usual Euclidian norm and $\mathbb{R}^n \times \mathbb{R}^m$ be equipped with ||(x, y)|| := ||x|| + ||y||, setting $X := \mathbb{R}^n \times \mathbb{R}$. *L* is called a *normal integrand* ([11], Definition 14.27) if its *epigraphical map* $x \Rightarrow \text{epi} L(x, \cdot) := \{(v, r) \in X \mid L(x, v) \leq r\}$ is closed-valued and measurable. For any normal integrand *L* its *domain map* $x \Rightarrow \text{dom } L(x, \cdot) := \{v \in \mathbb{R}^n \mid L(x, v) < +\infty\}$ is measurable and L(x, v) is lower semicontinuous in *v* and measurable in *x* ([11], Proposition 14.28). We recall also the Filippov's Theorem in the form we need it. To $\Phi : X \Rightarrow X$, one associates the differential inclusion

$$x'(t) \in \Phi(x(t)), \quad x(0) = 0.$$
 (5)

Proposition 1.2 ([2], Theorem 10.4.1). Let $y \in W^{1,1}([0, T]; X)$ be such that y(0) = 0 and assume that

 Φ is k-Lipschitz on X, and $t \mapsto d(y'(t), \Phi(y(t)))$ is integrable on [0, T].

Then there exists $x \in W^{1,1}([0, T]; X)$, which is a solution of (5) such that

$$||x(t) - y(t)|| \le \eta(t) \quad \forall t \in [0, T], \quad and \quad ||x'(t) - y'(t)|| \le k\eta(t) + \gamma(t) \quad a.e. \text{ in } [0, T],$$

where $\gamma(t) := d(\gamma'(t), \Phi(\gamma(t)) \text{ and } \eta(t) := e^{kt} \int_0^t \gamma(s) ds.$

2. Main result

Theorem 2.1. Let for $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ the following conditions hold:

- (a) L satisfies (2) with some measurable function $\Theta : \mathbb{R}^n \to \mathbb{R}_+$ that satisfies (3);
- (b) *L* is bounded on bounded sets of its domain;
- (c) the domain map $x \rightrightarrows D(x) := \text{dom } L(x, \cdot)$ has non-empty closed convex values;
- (d) the epigraphical map $x \rightrightarrows F(x) := epi L(x, \cdot)$ is closed-valued and k-Lipschitz.

Then, if $V < +\infty$ and $x \in W^{1,1}([0,T]; \mathbb{R}^n)$ is a solution of (1), it is necessarily Lipschitz continuous.

Before proving this result, we state some relations between the regularity of L, F and D.

Lemma 2.2. If F is k-Lipschitz continuous, then D is k-Lipschitz continuous and if, moreover, F is closed valued, then L is a normal integrand.

If *L* is Lipschitz on its domain, and its domain mapping *D* is Lipschitz, then the epigraphical map *F* is Lipschitz. The converse is not true. For example, $L : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as L(x, v) = 0 if $x \ge v \ge 0$ or $x \le v \le 0$, and L(x, v) = 1 otherwise, is discontinuous, while its epigraphical map *F* is 1-Lipschitz.

Proof of Theorem 2.1. Let $V < +\infty$ in (1). We consider the following problem

$$\overline{V} := \min\left\{\int_{0}^{T} \bar{L}(y(t), y'(t)) dt \mid y \in W^{1,1}([0, T]; \mathbb{R}^{n}), y(0) = 0\right\},$$
(7)

where $\bar{L}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is defined by $\bar{L}(x, v) := L(x, \Pi_{D(x)}v) + \Theta(v)(1 - \mathbb{1}_{D(x)}v) + e^{kT}d(v, D(x))$, and $\mathbb{1}$ stands for the characteristic function, d for the distance function, Π for the projection mapping. The schema of the proof consists in obtaining that $V = \overline{V}$, which implies that any minimizer of (1) is also a minimizer of (7), and then establishing that (7) has Lipschitz minimizers. We need the following:

Lemma 2.3. (A) the function $(x, v) \rightarrow d(v, D(x))$ is Lipschitz continuous;

(B) the mapping $(x, v) \rightarrow \prod_{D(x)} v$ is locally uniformly continuous;

(C) *L* is a Borel function.

Proof. (A) By (c) the sets D(x) are non-empty, closed and convex. Hence, the mapping $(x, v) \rightarrow \Pi_{D(x)}v$ is single-valued. Fix (x, v) and (x', v'). By Lipschitz continuity of D (see Lemma 2.2), there exists $y \in D(x)$ such that $||y - \Pi_{D(x')}v'|| \leq k ||x - x'||$. Hence,

$$d(v, D(x)) - d(v', D(x')) = \inf_{y \in D(x)} ||v - y|| - \inf_{y' \in D(x')} ||v' - y'|| \le ||v - y|| - ||v' - \Pi_{D(x')}v'||$$

$$\le ||v - \Pi_{D(x')}v'|| + ||\Pi_{D(x')}v' - y|| - ||v' - \Pi_{D(x')}v'||$$

$$\le ||\Pi_{D(x')}v' - y|| + ||v - v'|| \le \max\{k, 1\} ||(x, v) - (x', v')||,$$

which yields that the function $(x, v) \rightarrow d(v, D(x))$ is Lipschitz continuous.

(B) Fix x_0 , v_0 and $r > d(v_0, D(x_0))$. By Lipschitz continuity of $(x, v) \rightarrow d(v, D(x))$, there exists some neighbourhood $U \times V$ of (x_0, v_0) such that r > d(v, D(x)) for all $x \in U$, $v \in V$. Thanks to the properties of the projection onto a closed convex set, one can deduce that for any $x', x \in U$ and any $v, v' \in V$,

$$\|\Pi_{D(x)}v - \Pi_{D(x')}v'\| \leq [2rk\|x - x'\|]^{1/2} + \|v - v'\|,$$

and so the mapping $(x, v) \rightarrow \prod_{D(x)} v$ is locally uniformly continuous.

(C) On $\mathbb{R}^n \times \mathbb{R}^n$, let us define the functions $l(x, v) := L(x, \Pi_{D(x)}v), \theta(x, v) := \Theta(v)(1 - \mathbb{1}_{D(x)}v)$, and $q(x, v) := e^{kT}d(v, D(x))$. Thus, \overline{L} is their sum. We will show that they are all Borelians.

By the assumption (d) *L* is a normal integrand (see Lemma 2.2). Since $(x, v) \mapsto \Pi_{D(x)} v$ is continuous, it is a Carathéodory mapping (cf. [11]). As the function *l* is a composition of *L* and *f* defined as $f(x, v) = \Pi_{D(x)} v$, i.e. l(x, v) = L(x, f(x, v)), then *l* is a normal integrand ([11], Proposition 14.45). So, *l* is Borelian.

From (A) it follows the continuity of the function $(x, v) \rightarrow d(v, D(x))$, hence q is Borelian.

The function θ is a product of two functions, namely $\theta = \gamma . \tau$, where $\gamma(x, v) := \Theta(v)$, and $\tau(x, v) := 1 - \mathbb{1}_{D(x)}v = 1 - \mathbb{1}_{Gr D}(x, v)$. Since *D* is a measurable map (being a domain map of a normal integrand), hence Gr *D* is a measurable set ([11], Theorem 14.8). Then $\mathbb{1}_{Gr D}(x, v)$ is measurable, as well as, τ . The function γ is measurable since in (a) Θ is supposed to be measurable. The proof of the lemma is completed. \Box

Lemma 2.4. $V = \overline{V}$.

Proof. Since $L \ge \overline{L} \ge 0$, clearly $V \ge \overline{V}$. To establish the opposite inequality, let us observe that $V < +\infty$ implies $\overline{V} < +\infty$. Take any ε -solution $\overline{x}_{\varepsilon}$ of (7), i.e. $\overline{x}_{\varepsilon} \in W^{1,1}([0, T]; \mathbb{R}^n)$ with $\overline{x}_{\varepsilon}(0) = 0$ such that

$$\int_{0}^{1} \bar{L}(\bar{x}_{\varepsilon}(s), \bar{x}_{\varepsilon}'(s)) \, \mathrm{d}s \leqslant \overline{V} + \varepsilon.$$

For almost all $t \in [0, T]$ set $p_{\varepsilon}(t) := \prod_{D(\bar{x}_{\varepsilon}(t))} \bar{x}'_{\varepsilon}(t), h_{\varepsilon}(t) := d(\bar{x}'_{\varepsilon}(t), D(\bar{x}_{\varepsilon}(t))) = \|\bar{x}'_{\varepsilon}(t) - p_{\varepsilon}(t)\|, y_{\varepsilon}(t) := (\bar{x}_{\varepsilon}(t), \int_{0}^{t} L(\bar{x}_{\varepsilon}(s), p_{\varepsilon}(s)) \, ds)$. Define the multi-valued map $\Phi(x, r) := F(x)$ for all $(x, r) \in X$.

Claim. The function y_{ε} and the map Φ verify the conditions of Filippov's Theorem.

Proof of the claim. By assumption (d) F is closed-valued, so $F(x) = \operatorname{epi} L(x, \cdot)$ is a closed set and the map Φ has closed images. Again by (d) Φ is k-Lipschitz. We will show that $y_{\varepsilon} : [0, T] \to X$ is absolutely continuous. One can easily see that $t \to (\bar{x}_{\varepsilon}(t), p_{\varepsilon}(t))$ is measurable. By measurability of L (L is a Borel function by (d)) $t \to L(\bar{x}_{\varepsilon}(t), p_{\varepsilon}(t))$ is also measurable. Thus, $t \to y'_{\varepsilon}(t) = (\bar{x}'_{\varepsilon}(t), L(\bar{x}_{\varepsilon}(t), p_{\varepsilon}(t)))$ is measurable. To see that y'_{ε} is integrable, it suffices to observe that because of the integrability of \bar{x}'_{ε} ,

$$\int_{0}^{T} \left\| y_{\varepsilon}'(t) \right\| \mathrm{d}t = \int_{0}^{T} \left\| \bar{x}_{\varepsilon}'(t) \right\| \mathrm{d}t + \int_{0}^{T} L\left(\bar{x}_{\varepsilon}(t), p_{\varepsilon}(t) \right) \mathrm{d}t \leqslant \int_{0}^{T} \left\| \bar{x}_{\varepsilon}'(t) \right\| + \overline{V} + \varepsilon < \infty$$

Since y_{ε} is a continuous function and Φ is a closed-valued Lipschitz map, the map $t \Rightarrow \Phi(y_{\varepsilon}(t))$ is continuous, hence measurable, which together with measurability of $t \rightarrow y'_{\varepsilon}(t)$ imply that the closed-valued map $t \Rightarrow \Pi_{\Phi(y_{\varepsilon}(t))}y'_{\varepsilon}(t)$ is measurable. Hence, one can choose a measurable selection $g(t) \in \Pi_{\Phi(y_{\varepsilon}(t))}y'_{\varepsilon}(t)$ (see [11], Corollary 14.6). Then the function $\gamma_{\varepsilon}(t) := d(y'_{\varepsilon}(t), \Phi(y_{\varepsilon}(t))) = ||y'_{\varepsilon}(t) - g(t)||$ is measurable.

To establish the integrability of $\gamma_{\varepsilon}(t)$, let us note that $\gamma_{\varepsilon}(t) \leq h_{\varepsilon}(t)$. Since $(x, v) \to d(v, D(x))$ is Lipschitz by Lemma 2.3, since \bar{x}_{ε} is continuous and \bar{x}'_{ε} is measurable on [0, T], then $t \to h_{\varepsilon}(t) = d(\bar{x}'_{\varepsilon}(t), D(\bar{x}_{\varepsilon}(t)))$ is measurable. Let $s : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous selection of D (which exists according to Michael Selection Theorem (see e.g. [2])). One has that for any $t \in [0, T]$, $h_{\varepsilon}(t) = d(\bar{x}'_{\varepsilon}(t), D(\bar{x}_{\varepsilon}(t))) \leq \|\bar{x}'_{\varepsilon}(t) - s(\bar{x}_{\varepsilon}(t))\|$, hence, by the integrability of \bar{x}'_{ε} and by the continuity of $s \circ \bar{x}_{\varepsilon}$

$$\int_{0}^{T} h_{\varepsilon}(t) \, \mathrm{d}t \leq \int_{0}^{T} \left\| \bar{x}_{\varepsilon}'(t) - s(\bar{x}_{\varepsilon}(t)) \right\| \, \mathrm{d}t \leq \int_{0}^{T} \left\| \bar{x}_{\varepsilon}'(t) \right\| \, \mathrm{d}t + \int_{0}^{T} \left\| s(\bar{x}_{\varepsilon}(t)) \right\| \, \mathrm{d}t < \infty.$$

Consequently, $\gamma_{\varepsilon}(t)$ is integrable and the proof of the claim is completed. \Box

We continue the proof of Lemma 2.4 by applying the Filippov's Theorem for y_{ε} and Φ . We obtain the existence of a solution $(x_{\varepsilon}, r_{\varepsilon}) \in W^{1,1}([0, T]; X)$ of the differential inclusion

$$(x_{\varepsilon}'(t), r_{\varepsilon}'(t)) \in \Phi(x_{\varepsilon}(t), r_{\varepsilon}(t)), \quad (x_{\varepsilon}(0), r_{\varepsilon}(0)) = (0, 0)$$

so that for $t \in [0, T]$

such that for
$$t \in [0, T]$$

$$\left\|\left(x_{\varepsilon}(t),r_{\varepsilon}(t)\right)-y_{\varepsilon}(t)\right\| \leq e^{kt}\int_{0}^{t}\gamma_{\varepsilon}(s)\,\mathrm{d}s,\quad \text{and}\quad \left\|\left(x_{\varepsilon}'(t),r_{\varepsilon}'(t)\right)-y_{\varepsilon}'(t)\right\| \leq k\,e^{kt}\int_{0}^{t}\gamma_{\varepsilon}(s)\,\mathrm{d}s+\gamma_{\varepsilon}(t).$$

By the second inequality and using that $\gamma_{\varepsilon}(t) \leq h_{\varepsilon}(t)$, for $t \in [0, T]$ we obtain

$$r_{\varepsilon}'(t) - L(\bar{x}_{\varepsilon}(t), p_{\varepsilon}(t)) \leqslant k \operatorname{e}^{kt} \int_{0}^{t} h_{\varepsilon}(s) \, \mathrm{d}s + h_{\varepsilon}(t) \leqslant k \operatorname{e}^{kt} \int_{0}^{1} h_{\varepsilon}(s) \, \mathrm{d}s + h_{\varepsilon}(t) \quad \text{a.e. in } [0, T].$$

Since $(x'_{\varepsilon}(t), r'_{\varepsilon}(t)) \in \Phi(x_{\varepsilon}(t), r_{\varepsilon}(t))$, this means that $r'_{\varepsilon}(t) \ge L(x_{\varepsilon}(t), x'_{\varepsilon}(t))$, so

$$L(x_{\varepsilon}(t), x_{\varepsilon}'(t)) - L(\bar{x}_{\varepsilon}(t), p_{\varepsilon}(t)) \leq k e^{kt} \int_{0}^{1} h_{\varepsilon}(s) ds + h_{\varepsilon}(t) \quad \text{a.e. in } [0, T],$$

hence

$$\int_{0}^{T} L(x_{\varepsilon}(t), x_{\varepsilon}'(t)) dt \leq \int_{0}^{T} [L(\bar{x}_{\varepsilon}(t), p_{\varepsilon}(t)) + h_{\varepsilon}(t)] dt + \left[\int_{0}^{T} h_{\varepsilon}(t) dt\right] k \int_{0}^{T} e^{kt} dt \leq \int_{0}^{T} \bar{L}(\bar{x}_{\varepsilon}(t), \overline{x}_{\varepsilon}'(t)) dt$$

Thus, $V \leq \overline{V} + \varepsilon$. Passing $\varepsilon \downarrow 0$ we obtain $V \leq \overline{V}$, and Lemma 2.4 is proved. \Box

From Lemma 2.4 it follows that any solution of (1) is a solution of (7) and vice versa. Thus, to complete the proof of the theorem, we need to check that \overline{L} satisfies the hypotheses of Proposition 1.1.

 \overline{L} is a Borel function thanks to Lemma 2.3(C). \overline{L} satisfies (2) with Θ that comes from assumption (a) and hence Θ satisfies (3). \overline{L} satisfies also the condition (4), because for any $x_0 \in \mathbb{R}^n$ and any $\rho > 0$, \overline{L} is bounded on the set $B[x_0, \rho] \times B[0, \rho]$. To establish that, let us recall that \overline{L} is a sum of the functions l, θ and q, and fix any $\rho > 0$. By Lemma 2.3 the functions $(x, v) \to d(v, D(x))$ and q are continuous, hence they are bounded on $B[x_0, \rho] \times B[0, \rho]$. Consequently, the mapping $(x, v) \to \Pi_{D(x)}v$ is bounded on $B[x_0, \rho] \times B[0, \rho]$. Hence, thanks to assumption (b), l is bounded on $B[x_0, \rho] \times B[0, \rho]$. Take $(x, v) \in B[x_0, \rho] \times B[0, \rho]$. If $v \notin D(x)$, then $\theta(x, v) = 0$. If $v \in D(x)$, then $(x, v) \in dom L$ and $\theta(x, v) = \Theta(v) \leq L(x, v)$. So, θ is bounded on $B[x_0, \rho] \times B[0, \rho]$. Finally, as a sum of bounded functions, \overline{L} is bounded and we have proved that it satisfies the hypotheses of Proposition 1.1.

Hence, if $x(\cdot)$ is a solution of (1) then $x(\cdot)$ is a solution to (7) and we can apply Proposition 1.1 to \overline{L} to obtain that $x(\cdot)$ is Lipschitz continuous. The proof of Theorem 2.1 is then complete. \Box

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