# General formulas for the smoothed analysis of condition numbers 

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#### Abstract

We provide estimates on the volume of tubular neighborhoods around a subvariety $\Sigma$ of real projective space, intersected with a disk of radius $\sigma$. The bounds are in terms of $\sigma$, the dimension of the ambient space, and the degree of equations defining $\Sigma$. We use these bounds to obtain smoothed analysis estimates for some conic condition numbers. To cite this article: P. Bürgisser et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006). © 2006 Académie des sciences. Published by Elsevier SAS. All rights reserved.


## Résumé

Formules générales pour l'analyse régularisée des nombres de conditionnement. Nous donnons des estimations du volume de l'intersection des voisinages tubulaires autour d'une sous-variété $\Sigma$ de l'espace projectif réel avec un disque de rayon $\sigma$. Les bornes s'expriment en fonction de $\sigma$, de la dimension de l'espace ambiant, et du degré des équations définissant $\Sigma$. Nous utilisons ces bornes pour obtenir des estimations au sens de l'analyse régularisé pour des nombres de conditionnement coniques. Pour citer cet article : P. Bürgisser et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).
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## Version française abrégée

Dans l'analyse numérique de nombreux problèmes, le nombre de conditionnement $\mathscr{C}$ du problème occupe une position centrale. Le plus souvent $\mathscr{C}(a)$ peut être écrit comme (ou borné par) l'inverse relativisée de la distance entre la donnée $a$ et un ensemble $\Sigma$ de problèmes mal posés. Cette observation a été utilisée par Demmel [3] pour obtenir des bornes sur l'espérance mathématique de plusieurs nombres de conditionnement, l'idée essentielle étant de décrire la probabilité pour que $\mathscr{C}(a) \geqslant \frac{1}{\varepsilon}$ au moyen du volume du voisinage tubulaire $T(\Sigma, \varepsilon)=\left\{a \in S^{n} \mid d_{R}(a, \Sigma) \leqslant \varepsilon\right\}$, $d_{R}$ étant la distance de Riemann sur la sphère $S^{n}$. Mais, les résultats de Demmel reposent sur un travail non publié de A. Ocneanu datant de 1985. Apparemment, ce travail contient une borne supérieure sur le volume autour d'une variété réelle en termes de degré (cf. Théorème 4.3 de [3]). Le résultat principal de cette Note fournit de telles bornes.

[^0]De plus, nous considérons l'intersection du voisinage tubulaires avec un disque de rayon $\sigma$ dans l'espace projectif, et ce paramètre $\sigma$ apparaît dans l'expression de nos bornes. Plus précisément, nous démontrons le résultat suivant.

Théorème. Soit $W$ une variété algébrique réelle non vide, dans l'espace projectif $\mathbb{P}^{n}$, définie par un polynôme homogène de degré $d>0$. Alors, pour tout $a \in \mathbb{P}^{n}$ et pour tout $0<\sigma, \varepsilon \leqslant 1$,

$$
\frac{\operatorname{vol}_{n}\left(T_{\mathbb{P}^{n}}(W, \varepsilon) \cap B_{\mathbb{P}^{n}}(a, \sigma)\right)}{\operatorname{vol}_{n} B_{\mathbb{P}^{n}}(a, \sigma)} \leqslant 4 \sum_{k=1}^{n-1}\binom{n}{k}(2 d)^{k}\left(1+\frac{\varepsilon}{\sigma}\right)^{n-k}\left(\frac{\varepsilon}{\sigma}\right)^{k}+\frac{n \mathcal{O}_{n}}{\mathcal{O}_{n-1}}(2 d)^{n}\left(\frac{\varepsilon}{\sigma}\right)^{n}
$$

Ici, $B_{\mathbb{P}^{n}}(a, \sigma)=\left\{x \in \mathbb{P}^{n} \mid d_{\mathbb{P}^{n}}(x, a)<\sigma\right\}$ et $\mathcal{O}_{n}$ denote le volume de la sphère $S^{n}$.
Nous utilisons ensuite ce théorème pour obtenir des estimations au sens de l'analyse régularisée pour les nombres de conditionnement de la résolution des équations linéaires et du calcul de valeurs propres.

## 1. Introduction

A central feature in the numerical analysis of many computational problems is the condition number $\mathscr{C}$ of the problem. Given an input $a$, the condition number $\mathscr{C}(a)$ measures the extent to which small perturbations of that input affect the output. Classically, probabilistic analysis of condition numbers assumes a probability distribution on the set of data, and then takes two forms: bounds on the tail of the distribution of $\mathscr{C}(a)$ showing that it is unlikely that $\mathscr{C}(a)$ will be large, and bounds on the expected value of $\ln (\mathscr{C}(a))$ estimating the average loss of precision and average running time. The recently introduced smoothed analysis [9, §3] replaces showing that "it is unlikely that $\mathscr{C}(a)$ will be large" by showing that "for all $a$ and all slight random perturbations $\Delta a$, it is unlikely that $\mathscr{C}(a+\Delta a)$ will be large".

The goal of this Note is to give bounds for the smoothed analysis for a large class of condition numbers. Many times $\mathscr{C}(a)$ can be written as (or bounded by) the relativized inverse of the distance from the considered input $a$ to a set of ill-posed problems $\Sigma$. We say that $\mathscr{C}$ is a conic condition number in $\mathbb{R}^{n+1}$ if there exists a semi-algebraic cone $\Sigma \subset \mathbb{R}^{n+1}$ (the set of ill-posed inputs) such that, for all points $a \in \mathbb{R}^{n+1}$,

$$
\mathscr{C}(a)=\frac{\|a\|}{\operatorname{dist}(a, \Sigma)},
$$

where $\left\|\|\right.$ and dist are the norm and distance induced by a specific inner product $\langle$,$\rangle in \mathbb{R}^{n+1}$.
A well-known example of a conic condition number is $\kappa_{F}(A):=\|A\|_{F}\left\|A^{-1}\right\|$, where $A$ is a square matrix. The Condition Number Theorem of Eckart and Young [4] states that $\kappa_{F}(A)$ is conic when $\Sigma$ is the set of singular matrices. It is argued in [3] that for many problems the condition number can be bounded by a conic one.

Since $\Sigma$ is a cone, for all $z \in \mathbb{R} \backslash\{0\}, \mathscr{C}(a)=\mathscr{C}(z a)$. Hence, we may restrict to data $a$ in the real projective space $\mathbb{P}^{n}$, in which case the condition number takes the form

$$
\begin{equation*}
\mathscr{C}(a)=\frac{1}{d_{\mathbb{P}^{n}}(a, \Sigma)}, \tag{1}
\end{equation*}
$$

where $\Sigma$ is interpreted now as a subset of $\mathbb{P}^{n}$ and $d_{\mathbb{P}^{n}}$ denotes the projective distance (i.e., $d_{\mathbb{P}^{n}}(x, y)=\sin d_{R}(x, y)$, with $d_{R}(x, y)$ being the angle between $x$ and $\left.y\right)$. We denote by $B_{\mathbb{P}^{n}}(a, \sigma)$ the ball of radius $\sigma$ around $a$ with respect to the projective distance.

The kind of smoothed analysis we use here, which was introduced in [1], focuses on the following number

$$
\sup _{a \in \mathbb{P}^{n}} \underset{z \in B \mathbb{P}^{n}(a, \sigma)}{\mathbf{E}} f(z),
$$

where the expected value is taken with respect to the uniform distribution in $B_{\mathbb{P}^{n}}(a, \sigma)$. Note that when $\sigma=1$ (the diameter of $\mathbb{P}^{n}$ ) the expected value is independent of $a$ and we obtain the average-case analysis.

Our main result is the following:

Theorem 1.1. Let $\mathscr{C}$ be a conic condition number with set of ill-posed inputs $\Sigma \subseteq \mathbb{P}^{n}$. Assume $\Sigma$ is contained in the projective zero-set $\mathcal{Z}(f)$ of a homogeneous polynomial $f \in \mathbb{R}\left[X_{0}, \ldots, X_{n}\right]$ of degree $d$. Then, for all $\sigma \in(0,1]$ and $t \geqslant 1$, we have

$$
\sup _{a \in \mathbb{P}^{n}} \operatorname{Prob}_{z \in B_{\mathbb{P}}(a, \sigma)}(\mathscr{C}(z) \geqslant t) \leqslant 4 \sum_{k=1}^{n-1}\binom{n}{k}(2 d)^{k}\left(1+\frac{1}{t \sigma}\right)^{n-k}\left(\frac{1}{t \sigma}\right)^{k}+\frac{n \mathcal{O}_{n}}{\mathcal{O}_{n-1}}(2 d)^{n}\left(\frac{1}{t \sigma}\right)^{n}
$$

and

$$
\sup _{a \in \mathbb{P}^{n}} \underset{z \in B_{\mathbb{P}}(a, \sigma)}{\mathbf{E}}(\ln \mathscr{C}(z)) \leqslant 2 \ln n+2 \ln d+2 \ln \frac{1}{\sigma}+5.3,
$$

where $\mathcal{O}_{n}:=\operatorname{vol}_{n}\left(S^{n}\right)=\frac{2 \pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$ is the volume of the unit sphere $S^{n} \subseteq \mathbb{R}^{n+1}$.

## 2. Applications

We briefly describe how smoothed analysis estimates follow from Theorem 1.1 for the condition numbers of linear equation and eigenvalue problems.

### 2.1. Linear equation solving

The first application of our result is for the classical condition number $\kappa_{F}(A)$ for $p \times p$ matrices. In this case, $\Sigma$ is the set of singular matrices, which is defined by a single equation namely, $\operatorname{det}(A)=0$. Therefore, $d=p$, and we obtain

$$
\sup _{A \in \mathbb{P}^{2}-1} \underset{M \in B(A, \sigma)}{\mathbf{E}}\left(\ln \kappa_{F}(M)\right) \leqslant 6 \ln p+2 \ln \left(\frac{1}{\sigma}\right)+5.3 .
$$

### 2.2. Eigenvalue computations

The sensitivity of the eigenvalues of a matrix $A \in \mathbb{R}^{p \times p}$ under small perturbations is measured by a condition number $\kappa_{\text {eigen }}(A)$ which, according to Wilkinson [12], is bounded by

$$
\begin{equation*}
\kappa_{\text {eigen }}(A) \leqslant \frac{\sqrt{2}\|A\|_{F}}{\operatorname{dist}(A, \Sigma)} \tag{2}
\end{equation*}
$$

where $\Sigma$ is the set of matrices having multiple eigenvalues. This set is given by the equation $\operatorname{disc}\left(\chi_{A}\right)=0$ where $\operatorname{disc}\left(\chi_{\mathrm{A}}\right)$ is the discriminant of the characteristic polynomial $\chi_{\mathrm{A}}$ of $A$, a homogeneous polynomial in the entries of $A$ of degree $p^{2}-p$ (see [1, Proposition 3.4]). It follows that

$$
\sup _{A \in \mathbb{P}^{p^{2}-1}} \underset{M \in B(A, \sigma)}{\mathbf{E}}\left(\ln \kappa_{\text {eigen }}(M)\right) \leqslant 8 \ln p+2 \ln \frac{1}{\sigma}+6
$$

## 3. From probability to the volume of tubular neighborhoods

For a conic condition number $\mathscr{C}$ with set of ill-posed inputs $\Sigma$, we have

$$
\underset{z \in B_{\mathbb{P}^{n}}(a, \sigma)}{\operatorname{Prob}}\left\{\mathscr{C}(z) \geqslant \frac{1}{\varepsilon}\right\}=\underset{z \in B_{\mathbb{P}^{n}}(a, \sigma)}{\operatorname{Prob}}\left\{d_{\mathbb{P}^{n}}(z, \Sigma) \leqslant \varepsilon\right\}=\frac{\operatorname{vol}_{n}\left(T_{\mathbb{P}^{n}}(\Sigma, \varepsilon) \cap B_{\mathbb{P}^{n}}(a, \sigma)\right)}{\operatorname{vol}_{n} B_{\mathbb{P}^{n}}(a, \sigma)},
$$

where the $\varepsilon$-neighborhood $T_{\mathbb{P}^{n}}(\Sigma, \varepsilon)$ is the set of points having projective distance to $\Sigma$ less than $\varepsilon$. The first equation of Theorem 1.1 thus follows from the following statement:

Theorem 3.1. Let $W \subset \mathbb{P}^{n}$ be a nonempty real algebraic variety defined by a homogeneous polynomial of degree $d>0$. Let $a \in \mathbb{P}^{n}$ and let $0<\sigma, \varepsilon \leqslant 1$. Then

$$
\frac{\operatorname{vol}_{n}\left(T_{\mathbb{P}^{n}}(W, \varepsilon) \cap B_{\mathbb{P}^{n}}(a, \sigma)\right)}{\operatorname{vol}_{n} B_{\mathbb{P}^{n}}(a, \sigma)} \leqslant 4 \sum_{k=1}^{n-1}\binom{n}{k}(2 d)^{k}\left(1+\frac{\varepsilon}{\sigma}\right)^{n-k}\left(\frac{\varepsilon}{\sigma}\right)^{k}+\frac{n \mathcal{O}_{n}}{\mathcal{O}_{n-1}}(2 d)^{n}\left(\frac{\varepsilon}{\sigma}\right)^{n} .
$$

Bounds on the average case analysis based on a similar geometric result were given by Demmel [3]. His results are based on unpublished (and apparently unavailable) results by A. Ocneanu on the volume of tubes. One objective of this Note is to provide one result of this kind.

## 4. Outline of proof

We briefly outline the main ideas behind the proof of Theorem 3.1. The proof is based on two main ingredients: a bound on the volume of tubes around submanifolds of the sphere, in terms of integrals of curvature (based on Weyl's tube formula [11]), and degree-based bounds on these curvature integrals. We first deal with the case of smooth hypersurfaces of spheres, and use the angular distance $d_{R}$. We denote by $B_{S^{n}}(a, \varphi)$ the $\varphi$-ball around $a \in S^{n}$ with respect to $d_{R}$ and by $T_{S^{n}}(M, \alpha)$ the $\alpha$-neighborhood of a submanifold $M \subseteq S^{n}$ with respect to $d_{R}$. Note that $\operatorname{vol}_{n} B_{\mathbb{P}^{n}}(a, \sigma)=\operatorname{vol}_{n} B_{S^{n}}(a, \varphi)$, where $\sigma=\sin \varphi$.

Let $M$ be a compact oriented smooth hypersurface of $S^{n}$. The orientation corresponds to the choice of a unit normal vector field $v$ on $M$. Denote by $\kappa_{1}(x), \ldots, \kappa_{n-1}(x)$ the principal curvatures at $x \in M$ of the hypersurface $M$ with respect to the given orientation. For $1 \leqslant i \leqslant n-1$ we define the $i$ th curvature $K_{M, i}(x)$ of $M$ at $x$ as the $i$ th elementary symmetric polynomial in $\kappa_{1}(x), \ldots, \kappa_{n-1}(x)$, and we let $K_{M, 0}(x):=1$. Let $U$ be an open subset of $M$. We define the integrals $\mu_{i}(U)$ and $\left|\mu_{i}\right|(U)$ of the ith curvature and ith absolute curvature over $U$ as

$$
\begin{equation*}
\mu_{i}(U):=\int_{U} K_{M, i} \mathrm{~d} M, \quad\left|\mu_{i}\right|(U):=\int_{U}\left|K_{M, i}\right| \mathrm{d} M \tag{3}
\end{equation*}
$$

Clearly, $\left|\mu_{i}(U)\right| \leqslant\left|\mu_{i}\right|(U)$. Note that $\mu_{0}(U)=\left|\mu_{0}(U)\right|=\operatorname{vol}_{n-1}(U)$ and $\left|\mu_{i}\right|\left(S^{n-1}\right)=0$ for $i>0$. Also note that $\left|\mu_{i}\right|\left(U_{1}\right) \leqslant\left|\mu_{i}\right|\left(U_{2}\right)$ for $U_{1} \subseteq U_{2}$, while this is not true for $\mu_{i}$.

We define the $\alpha$-tube $T^{\perp}(U, \alpha)$ around $U$ by (compare Gray [5, p. 34])

$$
T^{\perp}(U, \alpha):=\left\{x \in S^{n} ; \text { there is a line segment in } S^{n} \text { of length }<\alpha \text { from } x \text { to } U \text { that intersects } U \text { orthogonally }\right\} .
$$

Clearly, $T^{\perp}(U, \alpha) \subseteq T_{S^{n}}(U, \alpha)$ and in general the inclusion is strict.
In a seminal article [11], Weyl derived a formula for the volume of $\alpha$-tubes around a compact submanifold of Euclidean space or a sphere, provided that $\alpha$ is small enough. Using methods similar to those used in [11] we prove the following proposition, which gives an upper bound on the volume of tubes around a hypersurface that holds for any $\alpha$ (compare with Gray [5, Theorem 8.4, (8.6), p. 162]):

Proposition 4.1. Let $M$ be a compact oriented smooth hypersurface of $S^{n}$ and $U$ be a nonempty open subset of $M$. Then, for all $0<\alpha \leqslant \pi / 2$,

$$
\operatorname{vol}_{n} T^{\perp}(U, \alpha) \leqslant 2 \sum_{i=0}^{n-1} J_{i+1}(\alpha)\left|\mu_{i}\right|(U)
$$

where $J_{i+1}(\alpha):=\int_{0}^{\alpha}(\sin \rho)^{i}(\cos \rho)^{n-i-1} \mathrm{~d} \rho$.
The next goal is thus to bound the integrals of absolute curvature $\left|\mu_{i}\right|(U)$ when $M$ is a smooth hypersurface defined as the zero-set of a polynomial of degree $d$. The keystone of the proof is the following proposition:

Proposition 4.2. Let $f \in \mathbb{R}\left[X_{0}, \ldots, X_{n}\right]$ be homogeneous of degree $d>0$ with a nonempty zero set $V \subset S^{n}$, and such that the derivative of $f: S^{n} \rightarrow \mathbb{R}$ does not vanish on $V$. Orient $V$ by the unit normal vector field defined by $v(x)=\|\operatorname{grad} f(x)\|^{-1} \operatorname{grad} f(x)$ for all $x \in V$. Let $a \in S^{n}$ and $0<\varphi \leqslant \pi / 2$. Then, for any $0 \leqslant i<n$,

$$
\left|\mu_{i}\right|\left(V \cap B_{S^{n}}(a, \varphi)\right) \leqslant 2\binom{n-1}{i} \mathcal{O}_{n-1} d^{i+1}(\sin \varphi)^{n-i-1}
$$

Sketch of proof. The proof is based on two steps. First, we bound $\left|\mu_{n-1}\right|(V)$ using methods similar to those for bounding the Betti-Numbers and Euler characteristic of real algebraic sets [7], in particular using Bézout's theorem. Ideas in [10, p. 410] have been useful for this purpose. We prove that

$$
\begin{equation*}
\left|\mu_{n-1}\right|(V)=\int_{V}\left|K_{V, n-1}\right| \mathrm{d} V \leqslant \mathcal{O}_{n-1} d^{n} \tag{4}
\end{equation*}
$$

Second, we use methods of integral geometry [8] to reduce the problem of bounding the $\mu_{i}(U)$ to the codimension-one situation of (4). We briefly describe the key result needed for this purpose.

The orthogonal group $G=\mathrm{O}(n+1)$ operates on $S^{n}$ in the natural way. We denote by $d g$ the invariant volume element on the compact Lie group $G$ normalized such that the volume of $G$ equals one. We interpret $S^{i+1}$ as a submanifold of $S^{n}$ for $i<n$, e.g., given by the equations $x_{i+1}=\cdots=x_{n}=0$. Let $M$ be a compact oriented smooth hypersurface of $S^{n}$. For almost all $g \in G$, the integral of the $i$ th (absolute) curvature of $M \cap g S^{i+1}$, considered as a smooth hypersurface of $g S^{i+1}$, is well defined and this is also the case for $U \cap g S^{i+1}$ with $U$ any open subset of $M$. The following special case of the principal kinematic formula of integral geometry for spheres $[6,8]$ holds (this was shown by Chern [2] in Euclidean space):

Lemma 4.3. Let $U$ be an open subset of a compact oriented smooth hypersurface $M$ of $S^{n}$. Then, for $0 \leqslant i<n-1$,

$$
\mu_{i}(U)=\mathcal{C}(n, i) \int_{G} \mu_{i}\left(U \cap g S^{i+1}\right) \mathrm{d} g
$$

where

$$
\mathcal{C}(n, i)=(n-i-1)\binom{n-1}{i} \frac{\mathcal{O}_{n-1} \mathcal{O}_{n}}{\mathcal{O}_{i} \mathcal{O}_{i+1} \mathcal{O}_{n-i-2}} .
$$

In the situation of Proposition 4.2, let $U_{+}$be the set of points of $U:=V \cap B_{S^{n}}(a, \varphi)$ where $K_{V, i}$ is $>0$ and similarly define $U_{-}$where $K_{V, i}$ is $<0$. Then $\left|\mu_{i}\right|(U)=\left|\mu_{i}\left(U_{+}\right)\right|+\left|\mu_{i}\left(U_{-}\right)\right|$. By (4) and the monotonicity of $\left|\mu_{i}\right|$, we have $\left|\mu_{i}\left(U_{+} \cap g S^{i+1}\right)\right| \leqslant\left|\mu_{i}\right|\left(U_{+} \cap g S^{i+1}\right) \leqslant\left|\mu_{i}\right|\left(V \cap g S^{i+1}\right) \leqslant \mathcal{O}_{i} d^{i+1}$. Hence Lemma 4.3 implies that

$$
\left|\mu_{i}\left(U_{+}\right)\right| \leqslant \mathcal{C}(n, i) \mathcal{O}_{i} d^{i+1} \underset{g \in G}{\operatorname{Prob}}\left\{B_{S^{n}}(g a, \varphi) \cap S^{i+1} \neq \emptyset\right\},
$$

and a similar inequality for $U_{-}$. The probability on the right-hand side can then be bounded so to yield Proposition 4.2.

Combining Propositions 4.1 and 4.2, with $U=V \cap B_{S^{n}}(a, \varphi)$, and using the estimate $J_{i}(\alpha) \leqslant(\sin \alpha)^{i} / i$ for $i<n$ and $J_{n}(\alpha) \leqslant \frac{\mathcal{O}_{n}}{2 \mathcal{O}_{n-1}}(\sin \alpha)^{n}$, we obtain the following bound on the volume of the tube around a patch of a smooth hypersurface of the sphere:

Proposition 4.4. Let $f \in \mathbb{R}\left[X_{0}, \ldots, X_{n}\right]$ be homogeneous of degree $d>0$ with zero set $V=\mathcal{Z}(f)$ in $S^{n}$. Assume that the derivative of $f$ does not vanish on $V$. Let $a \in S^{n}$, and $0<\alpha, \varphi \leqslant \frac{\pi}{2}$. Then

$$
\operatorname{vol}_{n} T^{\perp}\left(V \cap B_{S^{n}}(a, \varphi), \alpha\right) \leqslant 4 \frac{\mathcal{O}_{n-1}}{n} \sum_{k=1}^{n-1}\binom{n}{k} d^{k}(\sin \alpha)^{k}(\sin \varphi)^{n-k}+\mathcal{O}_{n} d^{n}(\sin \alpha)^{n}
$$

With this proposition at hand, we can prove Theorem 3.1: Set $\varepsilon:=\sin \alpha, \sigma:=\sin \varphi$ (we work in the setting of spheres). We have to remove the smoothness assumption in Proposition 4.4 and to estimate the volume of the $\alpha$-neighborhood instead of the $\alpha$-tube.

Assume $W=\mathcal{Z}(f)$ with $f$ homogeneous of degree $d$. Set $g=f^{2}$, so that $\operatorname{deg} g=2 d$ and $\mathcal{Z}(f)=\mathcal{Z}(g)$. Let $\delta>0$ be smaller than any positive critical value of the function $g: S^{n} \rightarrow \mathbb{R}$. Then $D_{\delta}:=\left\{x \in S^{n} \mid g(x) \leqslant \delta\right\}$ is a compact domain with smooth boundary

$$
\partial D_{\delta}=\left\{x \in S^{n} \mid g(x)=\delta\right\} .
$$

Indeed, the derivative of $g-\delta: S^{n} \rightarrow \mathbb{R}$ does not vanish on $\partial D_{\delta}$ (use $\sum_{i} x_{i} \partial_{i} g(x)=2 d \cdot g(x)$ ). Moreover, note that $W=\bigcap_{\delta>0} D_{\delta}$ and hence that $\lim _{\delta \rightarrow 0} \operatorname{vol}_{n} D_{\delta}=\operatorname{vol}_{n}(W)=0$, as $\operatorname{dim} W<n$.

We next claim that $T_{S^{n}}(W, \alpha) \subseteq D_{\delta} \cup T_{S^{n}}\left(\partial D_{\delta}, \alpha\right)$ for $0<\alpha \leqslant \pi / 2$. To see this, let $x \in T_{S^{n}}(W, \alpha) \backslash D_{\delta}$ and $\gamma:[0,1] \rightarrow S^{n}$ be a line segment of length less than $\alpha$ such that $\gamma(1)=x$ and $\gamma(0) \in W$. Consider the function $G:[0,1] \rightarrow \mathbb{R}$, defined by $G(t):=g(\gamma(t))$ for $t \in[0,1]$. By assumption, $G(1)>\delta$ and $G(0)=0$. Hence there exists $\tau \in(0,1)$ such that $G(\tau)=\delta$. Thus $\gamma(\tau) \in \partial D_{\delta}$ and hence $d_{R}\left(x, \partial D_{\delta}\right)<\alpha$, and our claim is established.

We next observe that

$$
T_{S^{n}}\left(\partial D_{\delta}, \alpha\right) \cap B_{S^{n}}(a, \varphi) \subseteq T^{\perp}\left(\partial D_{\delta} \cap B_{S^{n}}(a, \varphi+\alpha), \alpha\right) .
$$

Combining this observation with the above claim, we thus obtain

$$
T_{S^{n}}(W, \alpha) \cap B_{S^{n}}(a, \varphi) \subseteq D_{\delta} \cup T^{\perp}\left(\partial D_{\delta} \cap B_{S^{n}}(a, \varphi+\alpha), \alpha\right)
$$

Then we apply Proposition 4.4 to $V=\partial D_{\delta}=\mathcal{Z}\left(g-\delta\|x\|^{2 d}\right)$ intersected with the ball $B_{S^{n}}(a, \varphi+\alpha)$. Taking into account that $\operatorname{vol}_{n} B_{S^{n}}(a, \varphi) \geqslant \mathcal{O}_{n-1} \frac{\sigma^{n}}{n}$ and $\sin (\alpha+\varphi) \leqslant \varepsilon+\sigma$, we obtain

$$
\begin{aligned}
\frac{\operatorname{vol}_{n} T_{S^{n}}(W, \alpha) \cap B_{S^{n}}(a, \varphi)}{\operatorname{vol}_{n} B_{S^{n}}(a, \varphi)} & \leqslant \frac{\operatorname{vol}_{n} D_{\delta}}{\operatorname{vol}_{n} B_{S^{n}}(a, \varphi)}+\frac{\operatorname{vol}_{n} T^{\perp}\left(\partial D_{\delta} \cap B_{S^{n}}(a, \varphi+\alpha), \alpha\right)}{\operatorname{vol}_{n} B_{S^{n}}(a, \varphi)} \\
& \leqslant \frac{\operatorname{vol}_{n} D_{\delta}}{\operatorname{vol}_{n} B_{S^{n}}(a, \varphi)}+4 \sum_{k=1}^{n-1}\binom{n}{k}(2 d)^{k}\left(1+\frac{\varepsilon}{\sigma}\right)^{n-k}\left(\frac{\varepsilon}{\sigma}\right)^{k}+\frac{n \mathcal{O}_{n}}{\mathcal{O}_{n-1}}(2 d)^{n}\left(\frac{\varepsilon}{\sigma}\right)^{n} .
\end{aligned}
$$

Taking the limit as $\delta \rightarrow 0$ proves the assertion.

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