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Partial Differential Equations

Local gradient estimates of solutions to some conformally invariant fully nonlinear equations

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Abstract

We establish local gradient estimates to solutions of general conformally invariant fully nonlinear second order elliptic equations. ***To cite this article: Y.Y. Li, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Estimations locales du gradient des solutions pour quelques équations complètement non linéaires invariantes par transformation conforme. On démontre des estimations locales du gradient des solutions pour certaines équations elliptiques du second ordre, complètement non linéaires, invariantes par transformation conforme. ***Pour citer cet article : Y.Y. Li, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Version française abrégée

On suppose $\Gamma \subset \mathbb{R}^n$ est un cône ouvert convexe et symétrique de sommet 0 (= origine),

$$\Gamma_n := \{\lambda \mid \lambda_i > 0, 1 \leqslant i \leqslant n\} \subset \Gamma \subset \left\{\lambda \,\middle|\, \sum_{i=1}^n \lambda_i > 0\right\} =: \Gamma_1,$$

$$f \in C^1(\Gamma) \cap C^0(\overline{\Gamma}) \text{ est symétrique par rapport à } \lambda_i,$$

$$f \text{ est homogène de degré } 1,$$

$$f > 0,\ f_{\lambda_i} := \frac{\partial f}{\partial \lambda_i} > 0 \quad \text{dans } \Gamma, \quad f|_{\partial\Gamma} = 0,$$

$$\text{il existe } \delta > 0 \text{ tel que} \quad \sum_{i=1}^n f_{\lambda_i} \geqslant \delta \quad \text{dans } \Gamma.$$

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Soit (M, g) une variété riemannienne régulière de dimension $n \geqslant 3$. On considère le tenseur de Schouten

$$A_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)} g\right),$$

où Ric_g et R_g sont le tenseur de Ricci et la courbure scalaire. On denote par $\lambda(A_g) = (\lambda_1(A_g), \ldots, \lambda_n(A_g))$ les valeurs propres de A_g par rapport à g.

Théorème 1. *Soit u une solution positive de classe C^3 de l'équation*

$$f\big(\lambda(A_{u^{\frac{4}{n-2}} g})\big) = h, \quad \lambda(A_{u^{\frac{4}{n-2}} g}) \in \Gamma \text{ dans } B_{9r},$$

où B_{9r} est une boule géodésique dans M de rayon $9r$. Alors

$$\big\|\nabla(\log u)\big\|_g \leqslant C \quad \text{dans } B_r,$$

où $C > 0$ dépend seulement de (M, g), (f, Γ) et des bornes supérieures de $\sup_{B_{9r}} u$ et $\|h\|_{C^1(B_{9r})}$.

1. Introduction

We study local gradient estimates to solutions of conformally invariant fully nonlinear second order elliptic equations. Assume that

$$\Gamma \subset \mathbb{R}^n \text{ is an open convex symmetric cone with vertex at the origin,} \tag{1}$$

$$\Gamma_n := \{\lambda \mid \lambda_i > 0, 1 \leqslant i \leqslant n\} \subset \Gamma \subset \left\{\lambda \,\middle|\, \sum_{i=1}^n \lambda_i > 0\right\} =: \Gamma_1, \tag{2}$$

$$f \in C^1(\Gamma) \cap C^0(\overline{\Gamma}) \text{ is symmetric in } \lambda_i, \tag{3}$$

$$f \text{ is homogeneous of degree 1,} \tag{4}$$

$$f > 0, \ f_{\lambda_i} := \frac{\partial f}{\partial \lambda_i} > 0 \quad \text{in } \Gamma, \quad f|_{\partial\Gamma} = 0, \tag{5}$$

$$\sum_{i=1}^n f_{\lambda_i} \geqslant \delta, \quad \text{in } \Gamma \text{ for some } \delta > 0. \tag{6}$$

If (f, Γ) satisfies (1), (2), (4), (5) and

$$f \in C^2(\Gamma) \cap C^0(\overline{\Gamma}) \text{ is symmetric in } \lambda_i, \text{ and is concave in } \Gamma, \tag{7}$$

then (6) is automatically satisfied; see [14].

Examples of such (f, Γ) include those given by elementary symmetric functions. For $1 \leqslant k \leqslant n$ let $\sigma_k(\lambda) := \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \lambda_{i_1} \cdots \lambda_{i_k}$ be the k-th elementary symmetric function and let Γ_k be the connected component of $\{\lambda \in \mathbb{R}^n \mid \sigma_k(\lambda) > 0\}$ containing the positive cone Γ_n. Then $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$ satisfies all the above properties; see [1].

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geqslant 3$. We use i_0 and R_{ijkl} denote respectively the injectivity radius and the curvature tensor. Consider the Schouten tensor

$$A_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)} g\right),$$

where Ric_g and R_g denote respectively the Ricci tensor and the scalar curvature. We use $\lambda(A_g) = (\lambda_1(A_g), \ldots, \lambda_n(A_g))$ to denote the eigenvalues of A_g with respect to g.

Let $\hat{g} = u^{4/(n-2)} g$ be a conformal change of metrics, then

$$A_{\hat{g}} = -\frac{2}{n-2} u^{-1}\nabla^2 u + \frac{2n}{(n-2)^2} u^{-2} \nabla u \otimes \nabla u - \frac{2}{(n-2)^2} u^{-2} |\nabla u|^2 g + A_g,$$

where covariant derivatives are with respect to g. Let g_{flat} denote the Euclidean metric on $\mathbb{R}^n$, and let $g_1 = u^{4/(n-2)} g_{\text{flat}}$, then $A_{g_1} = u^{4/(n-2)} A^u_{ij}\, dx^i\, dx^j$ where

$$A^u := -\frac{2}{n-2}u^{-\frac{n+2}{n-2}}\nabla^2 u + \frac{2n}{(n-2)^2}u^{-\frac{2n}{n-2}}\nabla u \otimes \nabla u - \frac{2}{(n-2)^2}u^{-\frac{2n}{n-2}}|\nabla u|^2 I,$$

and I is the $n \times n$ identity matrix. In this case, $\lambda(A_{g_1}) = \lambda(A^u)$, where $\lambda(A^u)$ denotes the eigenvalues of the symmetric matrix A^u. We study

$$f\big(\lambda(A_{u^{\frac{4}{n-2}}g})\big) = h, \quad \lambda(A_{u^{\frac{4}{n-2}}g}) \in \Gamma. \tag{8}$$

Theorem 1. *Let* (M, g) *be as above and let* (f, Γ) *satisfy* (1)–(5) *and* (6). *For a geodesic ball* B_{9r} *in* M *of radius* $9r \leqslant \frac{1}{2}i_0$, *let* u *be a* C^3 *positive solution of* (8) *in* B_{9r}. *Then*

$$\big\|\nabla(\log u)\big\|_g \leqslant C \quad \text{in } B_r, \tag{9}$$

where C *is some positive constant depending only on* (f, Γ), *upper bounds of* i_0, $\sup_{B_{9r}} u$, $\|h\|_{C^1(B_{9r})}$ *and a bound of* R_{ijkl} *together with their first covariant derivatives.*

Remark 1. For $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$, the local gradient estimate (9) was established by Guan and Wang in [4]; see a related work [2] of Chang, Gursky and Yang. Such estimates were also studied in [7,5,13]. On locally conformally flat manifolds, 'semi-local' gradient estimates were established, and used, in [7,8] for (f, Γ) satisfying (1)–(3) and (5) via the method of moving spheres (or planes). In addition to the hypotheses in Theorem 8, if one assumes (7), estimate (9) was established by Chen in [3]. This latter result is a corollary of Theorem 1.3 in [11] and the proof of (1.39) in [7]; it has been observed independently by Wang in [16] that the result follows from the above mentioned Liouville theorem in [11]. The main point of Theorem 1 is that no concavity assumption is made on f.

2. A two step estimate of $|\nabla v_i|$

A subtlety of the local gradient estimate (9) is that the bound depends on an upper bound of u, but not on the upper bound of u^{-1}. Global estimates of $|\nabla u|$ allowing the dependence of an upper bound of both u and u^{-1} was given by Viaclovsky in [15]; see a related work [10]. One application of the local gradient estimate is for a rescaled sequence of solutions in the following situation: For solutions $\{u_i\}$ of (8) in a unit ball B_1 satisfying, for some constant $b > 0$ independent of i, $\sup_{B_1} u_i \leqslant b u_i(0) \to \infty$, consider

$$v_i(y) := \frac{1}{u_i(0)} v_i\left(\frac{y}{u_i(0)^{\frac{2}{n-2}}}\right).$$

One knows that

$$v_i(0) = 1, \quad \text{and} \quad v_i(y) \leqslant b \quad \forall\, |y| \leqslant u_i(0)^{\frac{2}{n-2}}, \tag{10}$$

and v_i satisfies the same equation with g replaced by the rescaled metric $g^{(i)}$. One would like to derive a bound of $|\nabla v_i|$ on $\{y \mid |y| < \beta\}$ for any fixed $\beta > 1$.

Some time ago the author arrived at the following idea: Try to establish the estimate of $|\nabla v_i|$ in two steps.

Step 1. To establish, for solutions u of (8) for general (f, Γ), local gradient estimates which depend on an upper bound of both u and u^{-1}.

Step 2. To establish, for solutions u of (8) in B_1 satisfying $u(0) = 1$, an estimate on B_δ of u^{-1} from above, which depends on an upper bound of u.

Once these two steps were achieved, the needed gradient bound for solutions $\{v_i\}$ satisfying (10) would follow. Aobing Li and the author then started to implement this idea. Step 1 for locally conformally flat manifolds is a special case of the 'semi-local' gradient estimate mentioned in Remark 1. We established Step 1 on general manifolds and for general (f, Γ):

Theorem 2. *([9]) Let* (M, g) *be as above and let* (f, Γ) *satisfy* (1)–(5) *and* (6). *For a geodesic ball* B_{9r} *in* M *of radius* $9r \leqslant \frac{1}{2}i_0$, *let* u *be a* C^3 *positive solution of* (8) *in* B_{9r} *satisfying, for some positive constants* $0 < a < b < \infty$, $a \leqslant u \leqslant b$ *on* B_{9r}. *Then* (9) *holds, where* C *is some positive constant depending only on* a, b, δ, *upper bounds of* i_0, $\|h\|_{C^1(B_{9r})}$ *and a bound of* R_{ijkl} *together with their first covariant derivatives.*

This result was extended to manifolds with boundary under prescribed mean curvature boundary conditions in [6], see Theorem 1.3 there. The method the author had in mind for Step 2 was to obtain, via Bernstein-type estimates, a bound on $|\nabla\Phi(u)| = |\Phi'(u)\nabla u|$ for an appropriate Φ. For instance, $|\nabla(u^\alpha)| \leqslant C$ for $\alpha < 0$ is weaker than $|\nabla \log u| \leqslant C$, and it becomes weaker when α is smaller. On the other hand, an estimate of $|\nabla(u^\alpha)|$ for any $\alpha < 0$ would yield an upper bound of u^{-1} near the origin. In principal, estimating $|\nabla(u^\alpha)|$ for very negative α should be easier than estimating $|\nabla \log u|$. However we encountered some difficulties in completing this step.

The author then took another path which requires establishing appropriate Liouville theorems for general degenerate conformally invariant equations

$$f\big(\lambda\big(A^u\big)\big) = 0, \quad \text{in } \mathbb{R}^n. \tag{11}$$

What is needed is to prove that any positive locally Lipschitz function u satisfying (11) in appropriate weak sense must be a constant. In [11], a notion of weak solutions, tailored for the application in local gradient estimates, was introduced. Such Liouville theorem for C^1_{loc} weak solutions of (11) is established there. My first impression was that weakening the regularity assumption from C^1_{loc} to $C^{0,1}_{\mathrm{loc}}$ (locally Lipschitz) is perhaps a subtle borderline issue whose solution would require some new ideas beyond those used in [11]. It turns out, to our surprise, that this only requires some modification of our proof of the Liouville theorem for C^1_{loc} weak solutions. Here is the improvement of Theorem 1.2 in [11]:

Theorem 3. *Let (f, Γ) satisfy* (1)–(3) *and* (5)*, and let u be a positive locally Lipschitz weak solution of* (11) *in the sense of Definition* 1.1 *in* [11]. *Then $u \equiv u(0)$.*

Remark 2. The conclusion of Theorem 3 still holds when replacing the locally Lipschitz property of u by $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $p > n$.

Theorem 3 follows from results in [11] and some slight improvement of the comparison principle Proposition 4.1 in [11]. Theorem 3 allows us to, using Theorem 2, first establish a local Hölder estimate of $\log u$ instead of the local gradient estimate of $\log u$. With the Hölder estimate of $\log u$, we then obtain the local gradient estimate of $\log u$ by another application of Theorem 2. The following problem looks reasonable and worthwhile to the author: Use the Bernstein-type estimates to complete the above mentioned Step 2, without any concavity assumption on f, by choosing appropriate Φ. The proof of the results in this Note can be found in [12].

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