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Complex Analysis

On the stability group of CR manifolds

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Abstract

For any essentially finite minimal real-analytic generic submanifold $M \subset \mathbb{C}^N$, $N \ge 2$, we show that for every point $p \in M$ the local real-analytic CR automorphisms of M fixing p can be parametrized real-analytically by their $\ell = \ell(p)$ jets at p. As an application, we derive a Lie group structure for the stability group Aut(M, p). We also show that the order $\ell = \ell(p)$ of the jet space in which the group Aut(M, p) embeds can be chosen to depend upper-semicontinuously on p. This yields that given any compact real-analytic minimal CR submanifold M in \mathbb{C}^N , there exists an integer k depending only on M such that for every point $p \in M$ local CR diffeomorphisms mapping a neighbourhood of p in M into another real-analytic CR submanifold in \mathbb{C}^N with the same CR dimension as that of M are uniquely determined by their k-jet at p. To cite this article: B. Lamel, N. Mir, C. R. Acad. Sci. Paris, Ser. I 343 (2006).

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Résumé

Sur le groupe d'isotropie des variétés CR. Pour toute sous-variété analytique réelle générique essentiellement finie et minimale $M \subset \mathbb{C}^N$, $N \ge 2$, nous montrons que pour tout point $p \in M$, les automorphismes CR locaux analytiques réels de M fixant p sont paramétrés analytiquement par leur $\ell = \ell(p)$ -jets en p. Comme application, nous obtenons une structure de groupe de Lie sur le groupe d'isotropie Aut(M, p). Nous montrons aussi que l'ordre $\ell = \ell(p)$ de l'espace des jets dans lequel le groupe Aut(M, p) se plonge peut être choisi de façon à ce que l'application $p \mapsto \ell(p)$ soit semi-continue supérieurement. En corollaire, nous obtenons qu'étant donnée toute sous-variété CR compacte analytique réelle et minimale $M \subset \mathbb{C}^N$, il existe un entier positif k, dépendant uniquement de M, tel que pour tout point $p \in M$ les difféomorphismes CR locaux envoyant un voisinage de p dans M sur toute autre sous-variété CR de \mathbb{C}^N de même dimension CR que celle de M sont uniquement déterminés par leur k-jet en p. *Pour citer cet article : B. Lamel, N. Mir, C. R. Acad. Sci. Paris, Ser. I 343 (2006).*

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1. Introduction and results

Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold and $p \in M$, $N \ge 2$. Throughout this note, we denote by Aut(M, p) the stability group of (M, p) i.e. the topological group of all local real-analytic CR automorphisms fixing

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the germ (M, p) equipped with the topology of uniform convergence on compact neighbourhoods of p. After the works of E. Cartan, Tanaka and Chern–Moser concerning the equivalence problem for real submanifolds in complex space, a number of interesting properties of the stability group of Levi-nondegenerate real-analytic hypersurfaces have been obtained by many mathematicians (see e.g. the surveys [13,3]).

The goal of this Note is to explore the possible generalization of these results to Levi-degenerate real-analytic hypersurfaces or CR manifolds of higher codimension. The investigation of such properties for Levi-degenerate submanifolds was first carried out in the recent work of Baouendi, Ebenfelt, Rothschild [1] and in the subsequent works [14,2]. For a certain class of Levi-degenerate real-analytic generic submanifolds of \mathbb{C}^N , it was shown in [1,14,2] that the local CR automorphisms of such CR manifolds depend analytically on their k-jets at a given point and for a certain integer k. Such a result allows one to obtain many interesting properties of the structure of the corresponding stability groups. However, the degeneracies considered in the above mentioned papers are of a special type and fail to hold for instance for arbitrary real-analytic hypersurfaces of \mathbb{C}^N containing no complex-analytic subvariety of positive dimension. Such a class of real hypersurfaces is of particular interest as observed in [8] (see also [5] for the general higher codimensional case) and it is well-known that boundaries of bounded domains with smooth real-analytic boundary are of this type (see [7]). In the case N = 2, Ebenfelt, Zaitsev and the first author have recently shown the analytic dependence of the local CR automorphisms of such a class of real hypersurfaces on their 2-jets (see [9]). In this note, we solve the jet-parametrization problem for the general class of essentially finite minimal real-analytic generic submanifolds of \mathbb{C}^N (of arbitrary codimension), which in particular include the above mentioned class of real hypersurfaces. Recall here that essential finiteness and minimality must be understood in the sense of [5] and [12] respectively. In what follows, for every integer k and every point $p \in \mathbb{C}^N$, we denote by $G_p^k(\mathbb{C}^N)$ the group of all k-jets at p of local biholomorphisms $H: (\mathbb{C}^N, p) \to (\mathbb{C}^N, p)$ and by $j_p^k H$ the k-jet of H at p.

Theorem 1. Let M be a real-analytic generic submanifold of \mathbb{C}^N that is essentially finite and minimal at each of its points. Then for every $p \in M$ there exists an integer ℓ_p , depending upper-semicontinuously on p, an open subset $\Omega \subset \mathbb{C}^N \times G_p^{\ell_p}(\mathbb{C}^N)$ and a real-analytic map $\Psi(Z, \Lambda) : \Omega \to \mathbb{C}^N$ holomorphic in the first factor, such that for any $H \in \operatorname{Aut}(M, p)$ the point $(p, j_p^{\ell_p} H) \in \Omega$ and the following identity holds:

$$H(Z) = \Psi(Z, j_p^{\ell_p} H)$$
 for all $Z \in M$ near p.

Moreover, the map Ψ has the following formal Taylor expansion:

$$\Psi(Z,\Lambda) = \sum \frac{P_{\alpha}(\Lambda,\Lambda)}{(Q(\Lambda^1,\bar{\Lambda}^1))^{s_{\alpha}}} (Z-p)^{\alpha},$$

where for every $\alpha \in \mathbb{N}^N$, s_α is a nonnegative integer, P_α and Q are polynomials in their arguments and Λ^1 denotes the linear part of the jet Λ .

As a direct application of Theorem 1 we have the following structure theorem on the stability group of any realanalytic generic essentially finite minimal submanifold of \mathbb{C}^N .

Theorem 2. Let M be a real-analytic generic submanifold of \mathbb{C}^N that is essentially finite and minimal at each of its points. Then for every $p \in M$ there exists an integer ℓ_p , depending upper-semicontinuously on p, such that the jet mapping $j_p^{\ell_p}$: Aut $(M, p) \to G_p^{\ell_p}(\mathbb{C}^N)$ is a continuous group homomorphism that is a homeomorphism onto a real-algebraic Lie subgroup of $G_p^{\ell_p}(\mathbb{C}^N)$.

Even in the case of real hypersurfaces of \mathbb{C}^N containing no complex-analytic subvariety of positive dimension with $N \ge 3$, the fact that the stability group $\operatorname{Aut}(M, p)$ is a Lie group was an open problem. Note that Theorem 2 is also new in the case N = 2 since the real-algebraicity of the stability group does not follow from [9]. Another novelty of Theorem 2 consists of providing an integer ℓ_p depending *upper-semicontinuously* on $p \in M$ for which the jet mapping $j_p^{\ell_p}$ is *merely injective*. (In the situation of Theorem 2 the existence of an integer k_p for which the jet mapping $j_p^{k_p}$: $\operatorname{Aut}(M, p) \to G_p^{k_p}(\mathbb{C}^N)$ is injective follows from the work [4], but there is no control of the dependence of k_p on the base point p in [4].) Such a dependence is crucial in order to get the following finite jet determination for compact CR submanifolds of \mathbb{C}^N .

Theorem 3. Let M be a compact real-analytic CR submanifold of \mathbb{C}^N minimal at each of its points. Then there is an integer k, depending only on M, such that for every $p \in M$ and for every real-analytic CR submanifold $M' \subset \mathbb{C}^N$ with the same CR dimension as that of M, smooth local CR diffeomorphisms mapping a neighbourhood of p in M into M' are uniquely determined by their k-jet at p.

Theorem 3 follows from the reflection principle proved in [5], the upper-semicontinuity of the map $p \mapsto \ell_p$ in Theorem 2 and the fact that compact real-analytic CR submanifolds of \mathbb{C}^N do not contain any germ of positive dimensional complex-analytic sets [7]. Note also that since compact real-analytic hypersurfaces are automatically minimal, Theorem 3 yields the following noteworthy result, which is new by itself.

Corollary 4. Let M be a compact real-analytic hypersurface in \mathbb{C}^N . Then there is an integer k, depending only on M, such that for every $p \in M$ and for every real-analytic hypersurface $M' \subset \mathbb{C}^N$, smooth local CR diffeomorphisms mapping a neighbourhood of p in M into M' are uniquely determined by their k-jet at p.

Finally let us mention the following application of Theorem 4 to proper holomorphic mappings of bounded domains with smooth real-analytic boundary, which can be viewed as a boundary version of H. Cartan's uniqueness theorem (see [6,10] on this matter).

Corollary 5. Let $\Omega \subset \mathbb{C}^N$ be a bounded domain with smooth real-analytic boundary. Then there exists an integer k, depending only on the boundary $\partial \Omega$, such that if $H : \Omega \to \Omega$ is a proper holomorphic mapping extending smoothly up to $\partial \Omega$ near some point $p \in \partial \Omega$ which satisfies $H(z) = z + o(|z - p|^k)$, then H is the identity mapping.

2. Main tools for the proofs

The proof of Theorem 1 is divided into two main steps. Firstly, due to the fact that we are dealing with generic submanifolds with possible strong Levi-degeneracies, we need to establish several results concerning the parametrization of solutions of a certain type of singular analytic systems of equations. One important result of this kind for non-linear analytic equations we wish to highlight in this note is given by the following theorem which seems to be of independent interest (see [11]).

Theorem 6. Let $A: (\mathbb{C}^n, 0) \to \mathbb{C}^n$ be a germ of a holomorphic map of generic rank n, X a complex manifold, and $b = b(z, \omega)$ be a \mathbb{C}^n -valued holomorphic map defined on an open neighbourhood of $\{0\} \times X \subset \mathbb{C}^n \times X$. Denote by $\mathsf{GL}_n(\mathbb{C})$ the group of invertible $n \times n$ matrices with complex coefficients. Then there exists a holomorphic map $\Gamma = \Gamma(z, \lambda, \omega) : \mathbb{C}^n \times \mathsf{GL}_n(\mathbb{C}) \times X \to \mathbb{C}^n$, defined on an open neighbourhood Ω of $\{0\} \times \mathsf{GL}_n(\mathbb{C}) \times X$, such that if $u: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a germ of a biholomorphism satisfying $A(u(z)) = b(z, \omega_0)$ for some $\omega_0 \in X$, then necessarily $u(z) = \Gamma(z, u'(0), \omega_0)$.

The fact that we can parametrize in Theorem 6 invertible solutions of the above type of analytic systems by their 1-jets at the origin is one of the reasons explaining why we are able to have an upper-semicontinuous dependence of the integer ℓ_p on $p \in M$ in Theorem 1.

The second step of the proof makes uses of the Segre set technique introduced by Baouendi, Ebenfelt and Rothschild [2]. Using the above mentioned result for singular analytic equations, we show that all elements of Aut(M, p) (as well as their jets) can be suitably parametrized by their jets at p when restricted to any Segre set. Then the obtained parametrization on a Segre set of a sufficiently high order yields as in [2] the desired parametrization Ψ thanks to the minimality assumption on M at p. Let us also mention that this proof works (and hence Theorem 1 holds) in fact for a class of real-analytic generic submanifolds of \mathbb{C}^N that is more general than the class of essentially finite minimal ones. For more details and complete proofs of Theorem 1 and of its consequences, we refer the reader to the article [11].

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