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# A domain embedding method for mixed boundary value problems 

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#### Abstract

We propose a domain embedding (fictitious domain) method for elliptic equations subject to mixed boundary conditions, and prove the sharp convergence rate. The theory provides a unified treatment for Dirichlet, Neumann, and Robin boundary conditions. To cite this article: S. Zhang, C. R. Acad. Sci. Paris, Ser. I 343 (2006). © 2006 Académie des sciences. Published by Elsevier SAS. All rights reserved.


## Résumé

Une méthode de domaine fictif pour des problèmes aux limites mixtes. Nous proposons une méthode de domaine fictif dans la résolution de problèmes elliptiques avec conditions aux limites mixtes. Nous établissons une estimation précise du taux de convergence de la solution d'un problème approché. La théorie donne un traitement unifié dans les cas de conditions aux limites de Dirichlet, de Neumann et de Robin. Pour citer cet article : S. Zhang, C. R. Acad. Sci. Paris, Ser. I 343 (2006).
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## 1. Method and theory

Let $\omega \subset \mathbb{R}^{2}$ be a bounded domain with Lipschitz boundary $\Gamma$ [1]. (The domain could be multiply connected.) For a given function $f$ on $\omega$, which we assume to be in $L^{p}(\omega)$ for some $p>1$, we seek a function $u$ on $\omega$ satisfying:

$$
-\Delta u=f \quad \text { in } \omega,
$$

with various boundary conditions on $\Gamma$. To make the presentation sufficiently general, we assume that $\Gamma$ is divided into three parts (one or two of which could be empty), such that $\Gamma=\bar{\Gamma}_{D} \cup \bar{\Gamma}_{N} \cup \bar{\Gamma}_{R}$, and on $\Gamma_{D}, \Gamma_{N}$, and $\Gamma_{R}$, homogeneous Dirichlet, Neumann, and Robin boundary conditions are imposed, respectively. That is,

$$
u=0 \quad \text { on } \Gamma_{D}, \quad \frac{\partial u}{\partial n}=0 \quad \text { on } \Gamma_{N}, \quad \frac{\partial u}{\partial n}+k u=0 \quad \text { on } \Gamma_{R} .
$$

[^0]

Fig. 1. A domain $\omega$ with mixed boundary condition embedded in a rectangular domain.

Here $n$ is the unit outward normal to $\Gamma$ and $k$ is a bounded and strictly positive function on $\Gamma_{R}$. As usual, we let $H^{1}(\omega)$ be the first order Sobolev space of functions on $\omega$ and $H_{D}^{1}(\omega)$ be the subspace whose functions vanish on $\Gamma_{D}$. The weak formulation of this boundary value problem is to find $u \in H_{D}^{1}(\omega)$ satisfying:

$$
\begin{equation*}
(\nabla u, \nabla v)_{\left[L^{2}(\omega)\right]^{2}}+(k u, v)_{L^{2}\left(\Gamma_{R}\right)}=\int_{\omega} f v \mathrm{~d} x \quad \forall v \in H_{D}^{1}(\omega), \tag{1}
\end{equation*}
$$

where $\nabla$ is the gradient operator, and the parentheses stand for the inner products in Hilbert spaces indicated by the subscripts. If $\Gamma_{D} \neq \emptyset$ or $\Gamma_{R} \neq \emptyset$, then (1) has a unique solution. If the problem is a pure Neumann problem, i.e., $\Gamma=\Gamma_{N}$, we need to assume $\int_{\omega} f \mathrm{~d} x=0$. Then $u$ is uniquely determined in the quotient space $H^{1}(\omega) / \mathbb{R}$.

The domain embedding method determines an approximation to $u$ by solving a boundary value problem on a larger domain $R \subset \mathbb{R}^{2}$ such that $\omega \subset R$. We let $\Omega=R \backslash \bar{\omega}$ be the fictitious domain. Corresponding to the splitting of the boundary $\Gamma=\bar{\Gamma}_{D} \cup \bar{\Gamma}_{N} \cup \bar{\Gamma}_{R}$, we divide $\Omega$ into three parts such that $\bar{\Omega}=\bar{\Omega}_{D} \cup \bar{\Omega}_{N} \cup \bar{\Omega}_{R}$, and $\partial \Omega_{D} \cap \Gamma=\bar{\Gamma}_{D}$, $\partial \Omega_{N} \cap \Gamma=\bar{\Gamma}_{N}$, and $\partial \Omega_{R} \cap \Gamma=\bar{\Gamma}_{R}$, see Fig. 1. We extend $f$ to a function $\bar{f}$ on $R$ with $\bar{f}=0$ on $\Omega$, and impose Dirichlet condition on $\partial R$. (Other boundary conditions are allowed, and sometimes, necessary.) For $\epsilon>0$, the domain embedding method determines $u^{\epsilon} \in H_{0}^{1}(R)$ such that for all $v \in H_{0}^{1}(R)$ :

$$
\begin{equation*}
\left(\nabla u^{\epsilon}, \nabla v\right)_{\left[L^{2}(\omega)\right]^{2}}+\left(k u^{\epsilon}, v\right)_{L^{2}\left(\Gamma_{R}\right)}+\epsilon^{-1}\left(\nabla u^{\epsilon}, \nabla v\right)_{\left[L^{2}\left(\Omega_{D}\right)\right]^{2}}+\epsilon\left(\nabla u^{\epsilon}, \nabla v\right)_{\left[L^{2}\left(\Omega \backslash \bar{\Omega}_{D}\right)\right]^{2}}=\int_{R} \bar{f} v \mathrm{~d} x . \tag{2}
\end{equation*}
$$

This is a well-defined problem on in $H_{0}^{1}(R)$.
Theorem 1.1. We assume that both $\Omega_{D}$ and $\omega \cup \Gamma_{D} \cup \Omega_{D}$ are Lipschitz domains. Furthermore, we assume that $\bar{\Gamma}_{N} \cup \bar{\Gamma}_{R} \subset R$ and $\Omega_{D}$ has no isolated and simply connected component. Then, as $\epsilon \rightarrow 0, u^{\epsilon}$ converges to a limit $u^{0} \in H_{0}^{1}(R)$. The limit satisfies

$$
\left.u^{0}\right|_{\Omega_{D}}=0, \quad \Delta\left(\left.u^{0}\right|_{\Omega \backslash \bar{\Omega}_{D}}\right)=0,\left.\quad u^{0}\right|_{\omega}=u, \quad \text { the solution of }(1) .
$$

And we have the sharp estimate that there exist positive constants $C_{1}$ and $C_{2}$ that may depend on $\omega, \Omega_{D}, \Omega_{N}, \Omega_{R}$, and $f$, but are independent of $\epsilon$, and such that

$$
C_{1} \epsilon \leqslant\left\|u^{\epsilon}-u^{0}\right\|_{H^{1}(\omega)}+\left\|u^{\epsilon}\right\|_{H^{1}\left(\Omega_{D}\right)} \leqslant\left\|u^{\epsilon}-u^{0}\right\|_{H^{1}(R)} \leqslant C_{2} \epsilon .
$$

There is an exceptional case in which $u^{\epsilon} \equiv u^{0}$. This occurs if and only if the solution of the original mixed boundary value problem on $\omega$ also satisfies homogeneous Dirichlet condition on $\Gamma_{N} \cup \Gamma_{R}$ and satisfies homogeneous Neumann condition on $\Gamma_{D} \backslash \partial R$.

Remark 1. The restriction $\bar{\Gamma}_{N} \cup \bar{\Gamma}_{R} \subset R$ is necessary because we have imposed homogeneous Dirichlet boundary condition on $\partial R$. If free condition is imposed on parts of $\partial R$ that touch $\Gamma_{N}$ or $\Gamma_{R}$, then this restriction can be removed.


Fig. 2. Solutions $u^{\epsilon}$ of the embedding equation and the limit $u^{0}$ (solid line), for $\epsilon=0.1$ (dashed line), 0.05 (dash-dot line), and 0.01 (dotted line).

Remark 2. If $\Omega_{D}$ has isolated and simply connected components, then $\left.u^{0}\right|_{\omega} \neq u$. In this case, a term of the form $\epsilon^{-1} \int_{\Omega_{0}} u^{\epsilon} v \mathrm{~d} x$ can be added to the left-hand side of (2) to correct the model. Here $\Omega_{0}$ is the union of the isolated and simply connected components of $\Omega_{D}$.

Remark 3. There is some freedom in the selection of $\Omega_{D}, \Omega_{N}$, and $\Omega_{R}$. But to ensure the convergence rate given above, both $\Omega_{D}$ and $\omega \cup \Gamma_{D} \cup \Omega_{D}$ should be Lipschitz domains.

Remark 4. If $\Omega=\Omega_{D}$ (i.e., $\Gamma=\Gamma_{D}$ ), or $\Omega=\Omega_{R}$, or $\Omega=\Omega_{N}$, Eq. (2) gives a domain embedding method for Dirichlet problem, Robin problem, or Neumann problem, respectively. Our sharp estimate for the former two problems seems new [2], and it improves the result on Neumann problem given in [3] and [4], where it was proved that

$$
\left|u^{\epsilon}-u^{0}\right|_{H^{1}(\omega)}=\mathrm{o}(\sqrt{\epsilon}) \quad \text { and } \quad \lim _{\epsilon \rightarrow 0}\left|u^{\epsilon}-u^{0}\right|_{H^{1}(\Omega)}=0
$$

As an illustration, we apply the method to an ordinary differential equation: Finding $u$ such that $-D^{2} u=1$ on the interval $(0,1), u(0)=0$, and $D u(1)=0$. We embed the interval $(0,1)$ in the larger interval $(-1,2)$, extend the righthand side function by 0 , and solve the following variant of the domain embedding method (2) to get $u^{\epsilon} \in H_{0}^{1}(-1,2)$.

$$
\epsilon^{-1}\left(D u^{\epsilon}, D v\right)_{L^{2}(-1,0)}+\left(D u^{\epsilon}, D v\right)_{L^{2}(0,1)}+\epsilon\left(D u^{\epsilon}, D v\right)_{L^{2}(1,2)}=(1, v)_{L^{2}(0,1)} \quad \forall v \in H_{0}^{1}(-1,2) .
$$

The solution of this equation is plotted in Fig. 2.

## 2. An abstract theory

Rather than proving the above theorem and remarks, we fit Eq. (2) in an abstract setting, and prove a more general result. The theorem and remarks then will be established by verifying the conditions enforced on the abstract problem. This abstract framework generalizes that in [5] under some conditions. In the following, we shall use the notation $\mathcal{P} \lesssim \mathcal{Q}$ which means that there exists a constant $C$ independent of $\epsilon$ such that $\mathcal{P} \leqslant C \mathcal{Q}$. The notation $\mathcal{P} \simeq \mathcal{Q}$ means $\mathcal{P} \lesssim \mathcal{Q}$ and $\mathcal{Q} \lesssim \mathcal{P}$.

Let $H, U, V$, and $W$ be Hilbert spaces, $A: H \rightarrow U$ and $B: H \rightarrow V$ be bounded linear operators, and $C: H \rightarrow W$ be a bounded linear operator with closed range. Furthermore, we assume that $B \times C: H \rightarrow V \times W$ also has closed range, which is defined as $(B \times C) v=(B v, C v) \in V \times W \forall v \in H$. We assume that

$$
\|A v\|_{U}+\|B v\|_{V}+\|C v\|_{W} \simeq\|v\|_{H} \quad \forall v \in H
$$

Thus the bilinear form

$$
(u, v)_{\mathcal{H}}:=(A u, A v)_{U}+(B u, B v)_{V}+(C u, C v)_{W}
$$

defines an equivalent inner product in $H$. Furnished with this new inner product, we denote the space by $\mathcal{H}$. For small but positive $\epsilon$, there exists a unique $u^{\epsilon} \in H$ such that

$$
\begin{equation*}
\epsilon\left(A u^{\epsilon}, A v\right)_{U}+\left(B u^{\epsilon}, B v\right)_{V}+\epsilon^{-1}\left(C u^{\epsilon}, C v\right)_{W}=\langle f, v\rangle, \quad \forall v \in H . \tag{3}
\end{equation*}
$$

Here $f \in H^{*}$ is a functional such that

$$
\begin{equation*}
\left.f\right|_{\operatorname{ker} B \cap \operatorname{ker} C}=0 . \tag{4}
\end{equation*}
$$

Since $B \times C$ has closed range in $V \times W$, we have:

$$
\begin{equation*}
\|B u\|_{V}+\|C u\|_{W} \simeq\|u\|_{H} \quad \forall u \in(\operatorname{ker} B \cap \operatorname{ker} C)_{\mathcal{H}}^{\perp} . \tag{5}
\end{equation*}
$$

So, $\|B u\|_{V} \simeq\|u\|_{H} \forall u \in \operatorname{ker} C \cap(\operatorname{ker} B \cap \operatorname{ker} C){ }_{\mathcal{\mathcal { H }}}$. Therefore, there exists a unique $u^{0} \in \operatorname{ker} C \cap(\operatorname{ker} B \cap \operatorname{ker} C)_{\mathcal{H}}^{\perp}$ such that

$$
\begin{equation*}
\left(B u^{0}, B v\right)_{V}=\langle f, v\rangle \quad \forall v \in \operatorname{ker} C . \tag{6}
\end{equation*}
$$

Since $C$ has closed range in $W$ and the functional $\langle f, v\rangle-\left(B u^{0}, B v\right)_{V}$ vanishes for any $v \in \operatorname{ker} C$, we have a unique $u_{0}^{1} \in(\operatorname{ker} C) \frac{\perp}{\mathcal{H}}$ such that

$$
\begin{equation*}
\left(C u_{0}^{1}, C v\right)_{W}=\langle f, v\rangle-\left(B u^{0}, B v\right)_{V} \quad \forall v \in H . \tag{7}
\end{equation*}
$$

It is easy to see that $\left\|A u^{0}\right\|_{U}+\left\|C u_{0}^{1}\right\|_{W} \lesssim\|f\|_{H^{*}}$.
Theorem 2.1. Under the assumptions that $\left.f\right|_{\text {ker } B \cap \operatorname{ker} C}=0, C$ has closed range in $W$, and $B \times C$ has closed range in $V \times W$, we have $\lim _{\epsilon \rightarrow 0} u^{\epsilon}=u^{0}$. The limit belongs to $\operatorname{ker} C \cap(\operatorname{ker} B \cap \operatorname{ker} C) \frac{\mathcal{H}}{\mathcal{H}}$, and is defined by (6). We have the estimate:

$$
\epsilon\left(\left\|C u_{0}^{1}\right\|_{W}+\kappa\left(\left\|C u_{0}^{1}\right\|_{W}\right)\left\|A u^{0}\right\|_{U}\right) \lesssim\left\|B\left(u^{\epsilon}-u^{0}\right)\right\|_{V}+\left\|C u^{\epsilon}\right\|_{W} \lesssim\left\|u^{\epsilon}-u^{0}\right\|_{H} \lesssim \epsilon\left(\left\|A u^{0}\right\|_{U}+\left\|C u_{0}^{1}\right\|_{W}\right)
$$

Here $u_{0}^{1} \in(\operatorname{ker} C)_{\mathcal{H}}^{\perp}$ is defined by (7). The function $\kappa$ is defined as $\kappa(x)=1$ if $x=0$ and $\kappa(x)=0$ otherwise. When $\epsilon \rightarrow 0$, we have either $u^{\epsilon} \equiv u^{0}$ or it converges to $u^{0}$ at the sharp rate of $\epsilon$. The former occurs if and only if $A u^{0}=0$ and $C u_{0}^{1}=0$.

Eq. (2) fits into (3) in an obvious manner. The definition of $\bar{f}$ ensures the condition (4). The assumption that both $\Omega_{D}$ and $\omega \cup \Gamma_{D} \cup \Omega_{D}$ are Lipschitz domains guarantees that the $C$ operator and $B \times C$ operator have closed range.

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