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Mathematical Problems in Mechanics

Existence of solutions for a dynamic Signorini's contact problem

María Teresa Cao, Peregrina Quintela

Departamento de Matemática Aplicada, Facultade de Matemáticas, Universidade de Santiago de Compostela, Santiago de Compostela 15782, Spain

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Abstract

The purpose of this work is to present an existence result for the dynamic frictionless contact problem between an elastic body and a rigid foundation. The proof is based on five fundamental steps: a discretization in time which leads to a discretized problem with unique solution; the construction of functions approximating a solution of the problem; the treatment of the contact condition by means of a Lagrange multiplier whose orthogonality properties allow us to get a priori estimates; the convergence of said functions and, finally, the pass to the limit obtaining a weak solution of the continuous problem. ***To cite this article: M.T. Cao, P. Quintela, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Un résultat d'existence à un problème dynamique de contact sans frottement en élasticité. Le but de ce travail est de présenter un résultat d'existence au problème dynamique de contact sans frottement entre un corps élastique et une fondation rigide. La preuve est basée sur cinq étapes fondamentales : une discrétisation en temps du problème qui mène à un problème à solution unique ; la construction de plusieurs séquences ; le traitement de la condition de contact au moyen d'un multiplicateur de Lagrange dont les propriétés d'orthogonalité nous permettent d'obtenir des estimations à priori et donc, obtenir la convergence des séquences ; finalement on passe à la limite pour obtenir une solution faible du problème continu. ***Pour citer cet article : M.T. Cao, P. Quintela, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Version française abrégée

Ce travail présente l'étude d'un problème dynamique de contact en élasticité linéaire entre un corps déformable et une fondation rigide. Pour modeler ce contact on utilise les conditions de Signorini. À notre connaissance, c'est la première fois jusqu'à présent que l'existence d'une solution à ce problème a été établie. Au cours de ces dernières années, de nombreux auteurs ont étudié une large gamme de problèmes de contact dynamiques avec différents matériaux et conditions aux limites. Par exemple, Duvaut et Lions [5] ont considéré des problèmes de contact quasi-statiques et dynamiques avec frottement pour des milieux élastiques et viscoélastiques. Les problèmes de contact dynamiques pour des milieux viscoélastiques ont été étudiés par plusieurs autres auteurs : Martins et Oden [10] dans le cas de

E-mail addresses: mcaorial@usc.es (M.T. Cao), mapere@usc.es (P. Quintela).

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compliance normale ; Cocou [3] pour les conditions Signorini avec une loi de frottement non local ; Cocou et Scarella [4] ont étendu plus tard ce résultat à un milieu fissuré ; Jaruseck [6] pour un problème de contact sans frottement avec des conditions de Signorini pour un milieu avec une mémoire singulière.

Pour l'équation d'ondes avec une condition unilatérale, Lebeau et Schatzman [8] ont établi l'existence et l'unicité d'une solution dans un demi-espace ; comme les auteurs eux-mêmes expliquent, leur méthode ne peut pas s'étendre aux domaines généraux. Les domaines bornés et réguliers ont été considérés par Kim [7] mais son travail ne peut pas être appliqué à l'élasticité.

On désigne par Ω un ouvert borné de $\mathbb{R}^n$, $n = 2, 3$ dont sa frontière Γ est $C^{1,1}$. Le problème qu'on considère est le suivante :

Problème (P). Trouver $(\boldsymbol{u}, \boldsymbol{\sigma})$ satisfaisant aux relations (1)–(6).

On retrouve, dans la relation (1), l'équation du mouvement, puis les conditions aux limites en forces imposées dans la relation (2) et en déplacements dans (3), ainsi que les conditions de contact unilatéral sans frottement dans (4) et (5) et les conditions initiales dans (6). $\boldsymbol{\sigma}(\boldsymbol{u})$ est le tenseur des contraintes qui est relié au tenseur des déformations $\boldsymbol{\varepsilon}(\boldsymbol{u})$ par la loi de Hooke $\boldsymbol{\sigma}(\boldsymbol{u}) = \Lambda^{-1}\boldsymbol{\varepsilon}(\boldsymbol{u})$, où Λ est le tenseur d'élasticité du quatrième ordre étaient indépendant du temps et vérifiant les propriétés usuelles de symétrie et d'ellipticité. Le terme $\boldsymbol{F}$ représente la densité de forces volumiques à laquelle est soumis le corps. $\boldsymbol{g}$ désigne les forces surfaciques imposées sur Γ_N. On suppose également que la densité ρ est constante. On considère aussi que les conditions initiales appartient à $[H^1(\Omega)]^n$ vérifiant div $\Lambda^{-1}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \in [L^2(\Omega)]^n$.

On obtient dans ce travail l'existence d'une solution au Problème (P) possédant la régularité suivante :

– $\boldsymbol{u} \in L^\infty(0, T; V_{\mathrm{ad}})$, $\dot{\boldsymbol{u}} \in L^\infty(0, T; [L^2(\Omega)]^n)$, et $\ddot{\boldsymbol{u}} \in D'(0, T; [L^2(\Omega)]^n)$,
– $\boldsymbol{\sigma}(\boldsymbol{u}) \in D'(0, T; E_{\mathrm{ad}}(\boldsymbol{g})) \cap L^\infty(0, T; [L^2(\Omega)]^{n^2})$,

où V_{ad} et E_{ad} sont les espaces fonctionnels définis dans (7) et (8) et le point au-dessus représente la dérivée par rapport au temps.

La preuve est basée sur cinq étapes fondamentales :

(1) Premièrement, nous faisons la discrétisation en temps du Problème (P) donnée par les équations (9)–(11) qui mène à une famille de problèmes à solution unique, $\boldsymbol{u}^i$, sur les intervalles $[t_{i-1}, t_i)$, $i = 1, \ldots, I$.
(2) Ensuite, nous construisons plusieurs séquences à partir des solutions $\boldsymbol{u}^i$ des problèmes discrétisés comme suit :

$$\boldsymbol{h}^I(t) = \boldsymbol{u}^{i-1} + \dot{\boldsymbol{u}}^{i-1}(t - t_{i-1}) + \frac{\ddot{\boldsymbol{u}}^{i-1} + \ddot{\boldsymbol{u}}^i}{4}(t - t_{i-1})^2, \quad \forall t \in [t_{i-1}, t_i); \quad \boldsymbol{h}^I(T) = \boldsymbol{u}^I, \quad \dot{\boldsymbol{h}}^I(T) = \dot{\boldsymbol{u}}^I,$$
$$\boldsymbol{h}^I_\star(t) = \frac{\boldsymbol{u}^i + \boldsymbol{u}^{i-1}}{2}, \quad \text{et} \quad \boldsymbol{u}^I_\star(t) = \boldsymbol{u}^i, \quad \forall t \in [t_{i-1}, t_i).$$

(3) Après, nous traitons la condition de contact au moyen d'un multiplicateur de Lagrange dont les propriétés d'orthogonalité nous permettent d'obtenir l'estimation à priori :

$$\rho \int_{t_{i-1}}^{t_i} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|\dot{\boldsymbol{h}}^I(t)\big\|^2_{[L^2(\Omega)]^n}\,\mathrm{d}t + \int_{t_{i-1}}^{t_i} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} a\big(\boldsymbol{h}^I(t), \boldsymbol{h}^I(t)\big)\,\mathrm{d}t \leqslant \int_\Omega \boldsymbol{\chi}^i : \boldsymbol{\varepsilon}\big(\boldsymbol{u}^i - \boldsymbol{u}^{i-1}\big)\,\mathrm{d}\boldsymbol{x}, \quad \forall t \in (t_{i-1}, t_i),$$

avec $a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \Lambda^{-1}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}$.
(4) Cette estimation nous permet de montrer la convergence faible des solutions approchées et que ses limites sont égales.
(5) Finalement on montre que cette limite est solution faible du problème continu.

1. Introduction

This Note studies a dynamic problem with contact between a linearly elastic deformable body and a rigid foundation. Said contact is modelled using Signorini conditions. To our knowledge, this is the first time to date that the existence of a solution for this problem has been achieved.

In recent years, many authors have studied a wide range of dynamic contact problems with various materials and boundary conditions. For example, in [5], Duvaut and Lions have considered quasistatic and dynamic contact problems with given friction for linearly elastic and viscoelastic bodies. Dynamic contact problems involving viscoelastic bodies have been studied by several other authors: Martins and Oden [10] in the case of normal compliance; Cocou [3] for Signorini conditions and nonlocal friction; Cocou and Scarella [4] later extended this result to a cracked body; Jarušeck in [6] for Signorini frictionless contact problem for materials with a singular memory.

For other problems such as wave equation with unilateral boundary conditions, Lebeau and Schatzman [8] have established the existence and uniqueness of solution for half-space domains. As the authors themselves explain, however, their method cannot be extended to general domains. Smooth bounded domains have been considered by Kim in [7] but his work cannot be applied to elasticity.

The Note is organized as follows. In Section 2 we will describe the model and the functional framework considered and Section 3 is devoted to the existence result and the main steps of its proof.

2. The model and functional framework

Let us consider an elastic solid with constant density ρ, initially occupying the bounded domain $\overline{\Omega} \subset \mathbb{R}^n$, $n = 2, 3$, of class $C^{1,1}$. The contact problem can be posed as follows:

Problem (P). Find $(\boldsymbol{u}, \boldsymbol{\sigma})$ verifying:

$$\rho \ddot{\boldsymbol{u}} - \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{F} \quad \text{in } \Omega \times (0, T), \tag{1}$$

$$\boldsymbol{\sigma} \boldsymbol{n} = \boldsymbol{g} \quad \text{on } \Gamma_N \times (0, T), \tag{2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D \times (0, T), \tag{3}$$

$$\boldsymbol{\sigma}_t = \boldsymbol{0}; \quad \sigma_n \leqslant 0; \qquad u_n \leqslant 0 \quad \text{on } \Gamma_C \times (0, T), \tag{4}$$

$$\sigma_n u_n = 0 \quad \text{on } \Gamma_C \times (0, T), \tag{5}$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0; \quad \dot{\boldsymbol{u}}(\boldsymbol{x}, 0) = \boldsymbol{u}_1 \quad \text{in } \Omega, \tag{6}$$

where $\boldsymbol{g} \in W^{2,\infty}(0, T; [L^2(\Gamma_N)]^n \cap [H^{-1/2}(\Gamma)]^n)$, $\boldsymbol{F} \in W^{2,\infty}(0, T; [L^2(\Omega)]^n)$, $\boldsymbol{\sigma}(\boldsymbol{u}) = \Lambda^{-1}\boldsymbol{\varepsilon}(\boldsymbol{u})$, Λ being the elasticity tensor assumed to be time independent, symmetric and coercive, $\boldsymbol{n}$ denotes the unit outward normal vector and $\operatorname{mes}(\Gamma_D) > 0$. The initial conditions $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ are assumed to belong to $[H^1(\Omega)]^n$ such that $\operatorname{div} \Lambda^{-1}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \in [L^2(\Omega)]^n$.

Let V be the space defined by $V = \{\boldsymbol{v} \in [H^1(\Omega)]^n;\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D\}$ and

$$V_{\text{ad}} = \{\boldsymbol{v} \in V;\ v_n \leqslant 0 \text{ on } \Gamma_C\}, \tag{7}$$

the closed and convex subset of admissible displacements.

We consider the space of the stress fields $X = \{\boldsymbol{\tau} = (\tau_{\alpha\beta}) \in [L^2(\Omega)]^{n^2};\ \tau_{\alpha\beta} = \tau_{\beta\alpha}\}$ and its subspace $E = \{\boldsymbol{\tau} \in X;\ \operatorname{div}(\boldsymbol{\tau}) \in [L^2(\Omega)]^n\}$, which is a Hilbert space. Given any function $\boldsymbol{f} \in [L^2(\Gamma_N)]^n$, we denote by $E_{\text{ad}}(\boldsymbol{f})$ the set of admissible stresses

$$E_{\text{ad}}(\boldsymbol{f}) = \{\boldsymbol{\tau} \in E;\ \boldsymbol{\tau}_t = \boldsymbol{0} \text{ and } \tau_n \leqslant 0 \text{ on } \Gamma_C;\ \boldsymbol{\tau}\boldsymbol{n} = \boldsymbol{f} \text{ on } \Gamma_N\}. \tag{8}$$

3. Existence of a solution

Theorem 3.1. *There exists a solution* $(\boldsymbol{u}, \boldsymbol{\sigma}(\boldsymbol{u}))$ *of Problem* (P) *verifying*:

- $\boldsymbol{u} \in L^\infty(0, T; V_{\text{ad}})$, $\dot{\boldsymbol{u}} \in L^\infty(0, T; [L^2(\Omega)]^n)$, *and* $\ddot{\boldsymbol{u}} \in D'(0, T; [L^2(\Omega)]^n)$.
- *The stress tensor* $\boldsymbol{\sigma}(\boldsymbol{u})$ *belongs to* $D'(0, T; E_{\text{ad}}(\boldsymbol{g})) \cap L^\infty(0, T; [L^2(\Omega)]^{n^2})$.

3.1. Sketch of the proof

We will present here only a sketch of the main steps of the proof. A complete proof of this theorem is given in [2].

3.1.1. Time discretization

Let us consider a regular partition of the time interval $[0, T]$ into I subintervals. Inspired on Newmark's method we propose the following approximation of Problem (P) at time $t = t_i$.

Problem (ADHP$^{\mathbf{i}}$). Find $\boldsymbol{u}^i \in V_{\mathrm{ad}}$, $\dot{\boldsymbol{u}}^i \in [H^1(\Omega)]^n$ and $\ddot{\boldsymbol{u}}^i \in [L^2(\Omega)]^n$ verifying the inequality

$$\int_\Omega \rho\left(\frac{\ddot{\boldsymbol{u}}^i + \ddot{\boldsymbol{u}}^{i-1}}{2}\right)\cdot\left(\boldsymbol{v}-\boldsymbol{u}^i\right)\mathrm{d}\boldsymbol{x} + \int_\Omega \Lambda^{-1}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{u}^i+\boldsymbol{u}^{i-1}}{2}\right) : \boldsymbol{\varepsilon}\left(\boldsymbol{v}-\boldsymbol{u}^i\right)\mathrm{d}\boldsymbol{x}$$
$$\geqslant \int_\Omega \boldsymbol{\chi}^i : \boldsymbol{\varepsilon}\left(\boldsymbol{v}-\boldsymbol{u}^i\right)\mathrm{d}\boldsymbol{x}, \quad \forall \boldsymbol{v}\in V_{\mathrm{ad}}, \tag{9}$$

where $\boldsymbol{\chi} \in W^{2,\infty}(0, T; [L^2(\Omega)]^{n^2})$ is a solution of

$$-\operatorname{div}\boldsymbol{\chi} = \boldsymbol{F} \quad \text{in } \Omega\times(0,T), \qquad \boldsymbol{\chi}\boldsymbol{n} = \boldsymbol{g} \quad \text{on } \Gamma_N\times(0,T), \qquad \boldsymbol{\chi}\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma_C\times(0,T), \tag{10}$$

$\boldsymbol{\chi}^i = \boldsymbol{\chi}(t_i)$, and the relation between the discretized fields of displacement, velocity and acceleration are framed in Newmark's method with parameters $\beta = 1/4$, $\gamma = 1/2$ as follows:

$$\boldsymbol{u}^i = \boldsymbol{u}^{i-1} + \Delta t\dot{\boldsymbol{u}}^{i-1} + \frac{\Delta t^2}{2}\frac{\ddot{\boldsymbol{u}}^i + \ddot{\boldsymbol{u}}^{i-1}}{2}, \qquad \dot{\boldsymbol{u}}^i = \dot{\boldsymbol{u}}^{i-1} + \Delta t\frac{\ddot{\boldsymbol{u}}^i + \ddot{\boldsymbol{u}}^{i-1}}{2}. \tag{11}$$

This allows us to rewrite the variational inequality (9) only in terms of the displacement. Thus, in order to approximate a solution of Problem (P) at each time t_i, $i = 1, 2, \ldots, I$, we consider the following algorithm:

– At the initial time, $\boldsymbol{u}^0 = \boldsymbol{u}(0) = \boldsymbol{u}_0$ and $\boldsymbol{u}^1 = \dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1$, while acceleration $\ddot{\boldsymbol{u}}^0$ is computed from the equilibrium equation as

$$\ddot{\boldsymbol{u}}^0 = \frac{1}{\rho}\left(\boldsymbol{F}^0 + \operatorname{div}\Lambda^{-1}\boldsymbol{\varepsilon}\left(\boldsymbol{u}^0\right)\right),$$

with $\boldsymbol{F}^0 = \boldsymbol{F}(0)$.

– For each time step t_i, given $\boldsymbol{u}^{i-1}$, $\dot{\boldsymbol{u}}^{i-1}$ and $\ddot{\boldsymbol{u}}^{i-1}$ we obtain $\boldsymbol{u}^i$ as the solution of the variational problem:

Problem (DHP$^{\mathbf{i}}$). Find $\boldsymbol{u}^i \in V_{\mathrm{ad}}$ such that:

$$\int_\Omega \rho\boldsymbol{u}^i\cdot\left(\boldsymbol{v}-\boldsymbol{u}^i\right)\mathrm{d}\boldsymbol{x} + \frac{\Delta t^2}{4}\int_\Omega \Lambda^{-1}\boldsymbol{\varepsilon}\left(\boldsymbol{u}^i\right):\boldsymbol{\varepsilon}\left(\boldsymbol{v}-\boldsymbol{u}^i\right)\mathrm{d}\boldsymbol{x} \geqslant \int_\Omega \rho\left[\boldsymbol{u}^{i-1} + \Delta t\dot{\boldsymbol{u}}^{i-1}\right]\cdot\left(\boldsymbol{v}-\boldsymbol{u}^i\right)\mathrm{d}\boldsymbol{x}$$
$$-\frac{\Delta t^2}{4}\int_\Omega \Lambda^{-1}\boldsymbol{\varepsilon}\left(\boldsymbol{u}^{i-1}\right):\boldsymbol{\varepsilon}\left(\boldsymbol{v}-\boldsymbol{u}^i\right)\mathrm{d}\boldsymbol{x} + \frac{\Delta t^2}{2}\int_\Omega \boldsymbol{\chi}^i:\boldsymbol{\varepsilon}\left(\boldsymbol{v}-\boldsymbol{u}^i\right)\mathrm{d}\boldsymbol{x}, \quad \forall\boldsymbol{v}\in V_{\mathrm{ad}}. \tag{12}$$

– Next, given $\boldsymbol{u}^{i-1}$, $\dot{\boldsymbol{u}}^{i-1}$, $\ddot{\boldsymbol{u}}^{i-1}$ and $\boldsymbol{u}^i$, we obtain $\ddot{\boldsymbol{u}}^i$ and $\dot{\boldsymbol{u}}^i$ using relations (11).

Note that if $\boldsymbol{u}^{i-1} \in V_{\mathrm{ad}}$, $\dot{\boldsymbol{u}}^{i-1} \in [H^1(\Omega)]^n$ and $\ddot{\boldsymbol{u}}^{i-1} \in [L^2(\Omega)]^n$, then, there exists a unique solution $\boldsymbol{u}^i$ of Problem (DHP$^{\mathrm{i}}$), and thanks to (11), $\dot{\boldsymbol{u}}^i \in [H^1(\Omega)]^n$, $\ddot{\boldsymbol{u}}^i \in [L^2(\Omega)]^n$, and $\ddot{\boldsymbol{u}}^i + \ddot{\boldsymbol{u}}^{i-1} \in [H^1(\Omega)]^n$.

It is also straightforward to prove that if $\boldsymbol{u}^i$, $\dot{\boldsymbol{u}}^i$ and $\ddot{\boldsymbol{u}}^i$ are solution of Problem (ADHP$^{\mathrm{i}}$), then, $\boldsymbol{u}^i$ is solution of Problem (DHP$^{\mathrm{i}}$) and $\boldsymbol{u}^i$, $\dot{\boldsymbol{u}}^i$ and $\ddot{\boldsymbol{u}}^i$ verify (11). The reciprocal result is also true.

The solution $(\boldsymbol{u}^i, \dot{\boldsymbol{u}}^i, \ddot{\boldsymbol{u}}^i)$ of Problem (ADHP$^{\mathrm{i}}$) has the following property: If we denote by

$$\boldsymbol{\sigma}\left(\frac{\boldsymbol{u}^i+\boldsymbol{u}^{i-1}}{2}\right) = \Lambda^{-1}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{u}^i+\boldsymbol{u}^{i-1}}{2}\right),$$

then $\boldsymbol{\sigma}(\frac{\boldsymbol{u}^i+\boldsymbol{u}^{i-1}}{2}) \in E_{\mathrm{ad}}(\boldsymbol{g}^i)$ and the following expressions hold:

$$\rho\frac{\ddot{\boldsymbol{u}}^i+\ddot{\boldsymbol{u}}^{i-1}}{2} - \operatorname{div}\boldsymbol{\sigma}\left(\frac{\boldsymbol{u}^i+\boldsymbol{u}^{i-1}}{2}\right) = \boldsymbol{F}^i \quad \text{in } \Omega; \qquad \sigma_n\left(\frac{\boldsymbol{u}^i+\boldsymbol{u}^{i-1}}{2}\right)u_n^i = 0 \quad \text{on } \Gamma_C. \tag{13}$$

3.1.2. Approximated solutions

In this step, several sequences are constructed from the solution of the approximated Problems (ADHP$^{\mathrm{i}}$), $1 \leqslant i \leqslant I$, when $I \to +\infty$. To do that, let us take the following functions:

$$\boldsymbol{h}^I(t) = \boldsymbol{u}^{i-1} + \dot{\boldsymbol{u}}^{i-1}(t - t_{i-1}) + \frac{\ddot{\boldsymbol{u}}^{i-1} + \ddot{\boldsymbol{u}}^i}{4}(t - t_{i-1})^2, \quad \forall t \in [t_{i-1}, t_i); \quad \boldsymbol{h}^I(T) = \boldsymbol{u}^I, \quad \dot{\boldsymbol{h}}^I(T) = \dot{\boldsymbol{u}}^I, \tag{14}$$

$$\boldsymbol{h}^I_\star(t) = \frac{\boldsymbol{u}^i + \boldsymbol{u}^{i-1}}{2}, \quad \text{and} \quad \boldsymbol{u}^I_\star(t) = \boldsymbol{u}^i, \quad \forall t \in [t_{i-1}, t_i). \tag{15}$$

It can be seen that $\boldsymbol{h}^I(t) \in C^1(0, T; [H^1(\Omega)]^n)$ and $\boldsymbol{h}^I$ is C^2 at each subinterval (t_{i-1}, t_i).

3.1.3. A priori estimates

To obtain a priori estimates the following lemma, which can be proved by using subdifferential techniques and maximal monotone operators theory, is essential.

Lemma 3.2. *Let $\boldsymbol{u}^i, \dot{\boldsymbol{u}}^i, \ddot{\boldsymbol{u}}^i$ be the solution of problem* (ADHP$^{\mathrm{i}}$). *Then, it is solution of*

$$\int_\Omega \rho\left(\frac{\ddot{\boldsymbol{u}}^i + \ddot{\boldsymbol{u}}^{i-1}}{2}\right) \cdot \boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \Lambda^{-1}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{u}^i + \boldsymbol{u}^{i-1}}{2}\right) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{\chi}^i : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} - \int_{\Gamma_C} p^i v_n\,\mathrm{d}\Gamma, \quad \forall \boldsymbol{v} \in V, \tag{16}$$

p^i being the solution of the nonlinear equation

$$p^i = \frac{1}{\lambda_c}\left[u^i_{n|\Gamma_C} + \lambda_c p^i - \Pi_P\left(u^i_{n|\Gamma_C} + \lambda_c p^i\right)\right], \tag{17}$$

where $\lambda_c \geqslant 0$ and Π_P is the projection of $L^2(\Gamma_C)$ onto $P = \{q \in L^2(\Gamma_C);\ q \leqslant 0$ c.p.d. on $\Gamma_C\}$.

The reciprocal result is also true.

As a consequence, $\int_{\Gamma_C} p^i(u^i_n - v_n)\,\mathrm{d}\Gamma \geqslant 0$ for all $\boldsymbol{v} \in V_{\mathrm{ad}}$ and the following a priori estimate holds:

$$\rho \int_{t_{i-1}}^{t_i} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\boldsymbol{h}}^I(t)\|^2_{[L^2(\Omega)]^n}\,\mathrm{d}t + \int_{t_{i-1}}^{t_i} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}a\big(\boldsymbol{h}^I(t), \boldsymbol{h}^I(t)\big)\,\mathrm{d}t \leqslant \int_\Omega \boldsymbol{\chi}^i : \boldsymbol{\varepsilon}\big(\boldsymbol{u}^i - \boldsymbol{u}^{i-1}\big)\,\mathrm{d}\boldsymbol{x}, \quad \forall t \in (t_{i-1}, t_i), \tag{18}$$

with $a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \Lambda^{-1}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}$.

3.1.4. Convergence of approximate solutions

From (18) and given that $\boldsymbol{h}^I_\star$ and $\boldsymbol{u}^I_\star$ can be written in terms of $\boldsymbol{h}^I(t)$, one can easily conclude that

- $\|\boldsymbol{h}^I(t_k)\|_{[H^1(\Omega)]^n}$ is bounded by a constant independent of I and k, $0 \leqslant k \leqslant I$,
- $\boldsymbol{h}^I$ and $\dot{\boldsymbol{h}}^I$ are bounded in $L^\infty(0, T; [L^2(\Omega)]^n)$ by a constant independent of I,
- $\boldsymbol{h}^I_\star$ and $\dot{\boldsymbol{u}}^I_\star$ are bounded in $L^\infty(0, T; [H^1(\Omega)]^n)$ by a constant independent of I,

and there exists $\boldsymbol{u} \in L^\infty(0, T; [H^1(\Omega)]^n)$ such that:

$$\boldsymbol{h}^I \rightharpoonup \boldsymbol{u} \quad \text{and} \quad \dot{\boldsymbol{h}}^I \rightharpoonup \dot{\boldsymbol{u}} \quad \text{weak * in } L^\infty\big(0, T; \big[L^2(\Omega)\big]^n\big),$$
$$\boldsymbol{h}^I_\star \rightharpoonup \boldsymbol{u} \quad \text{and} \quad \boldsymbol{u}^I_\star \rightharpoonup \boldsymbol{u} \quad \text{weak * in } L^\infty\big(0, T; \big[H^1(\Omega)\big]^n\big).$$

3.1.5. The limit $\boldsymbol{u}$ is solution of Problem (P)

Thanks to the definition of $\boldsymbol{h}^I$ and $\boldsymbol{h}^I_\star$ we get that

$$\int_0^T\!\!\int_\Omega \rho\ddot{\boldsymbol{h}}^I \cdot \boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^T\!\!\int_\Omega \Lambda^{-1}\boldsymbol{\varepsilon}\big(\boldsymbol{h}^I_\star\big) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \geqslant \int_0^T\!\!\int_\Omega \boldsymbol{\chi}^I_\star : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t, \quad \forall \boldsymbol{v} \in L^\infty(0, T; V_{\mathrm{ad}}), \tag{19}$$

from which we can recover Eqs. (1)–(4). In order to prove that $\boldsymbol{u}$ verifies (5) we follow the work of Barral and Quintela [1] and finally, following similar techniques to those in Lions [9] we prove that $\boldsymbol{u}$ verifies the initial conditions (6).

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