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Mathematical Problems in Mechanics

The Reissner–Mindlin plate theory via Γ-convergence

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Abstract

We obtain the energy functional of Reissner–Mindlin plates as the Γ-limit of a family of three-dimensional energy functionals within the framework of second-order linear elasticity. The choice of the family of functionals, as well as of the candidate limiting functional, is guided by a formal scaling argument. ***To cite this article: R. Paroni et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

La théorie des plaques de Reissner–Mindlin via Γ-convergence. Nous obtenons la fonctionnelle de l'énergie de plaques de Reissner–Mindlin comme la Γ-limite d'une famille de fonctionnelles d'énergie tri-dimensionnelles dans le cadre de l'élasticité linéaire du second ordre. Les choix de la famille de fonctionnelles et de la fonctionnelle limite sont suggérés par l'étude formelle des mises à l'échelle. ***Pour citer cet article : R. Paroni et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

1. Introduction

To derive the equations of thin structures from three-dimensional elasticity, one has to set up some limit process when thickness tends to zero. The immense literature on the subject may be sorted out according to the approach to the thickness limit. Three such approaches have been successfully proposed: *asymptotic analysis*, *variational convergence*, and *formal scaling*, the latter as an evolution of the *method of internal constraints*. Whatever the approach, if classic elasticity is selected as parent three-dimensional theory, then the induced lower-dimensional models have the mathematical form of Kirchhoff–Love's for plates and Bernoulli–Navier's for rods, that is to say, models of thin structures that can bend, but cannot shear. Thus, all approaches seem to be equally faulty, because they cannot capture thinness while preserving shear ability. This remarkable coincidence of failures indicates the need for a common change in format. Now, inspection of standard engineering derivations of the equations of shearable thin structures, makes clear that each of the kinematical *Ansätze* on which those derivations are based is in fact equivalent to a specific set of internal constraints, on both the first and the second gradient of admissible displacement fields: the first-order constraints are incompatible with the assumption of isotropicity; the second-order constraints call for stored-energy

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functions including second-gradient terms. Building on these observations in the light of the method of formal scaling, it can be shown that, *if the material is presumed to be transversely isotropic*, theories of either unshearable or shearable thin structures can be obtained; and that, *if the stored-energy density features certain second-gradient terms*, then shearable structure models obtain, including Reissner–Mindlin's plates and Timoshenko's rods [3]. These results prompt a conjecture: were the same parent theory chosen as in [3], either asymptotic analysis or variational convergence would also give those structure theories a precise limit position, this time, however, within a carefully stated functional setting. In this writing, exploiting the guidance offered by the formal scaling argument given in [3], we prove that such a conjecture holds true, in that we give the energy functional of Reissner–Mindlin plates the status of Γ-limit of a suitable family of three-dimensional energy functionals within the framework of second-order linear elasticity.

2. Formal scaling

Following [3], we pick a second-order elastic material whose stored-energy density is

$$\sigma\big(\nabla\mathbf{u},\nabla^{(2)}\mathbf{u}\big)=\sigma^{(1)}(\nabla\mathbf{u})+\sigma^{(2)}\big(\nabla^{(2)}\mathbf{u}\big),$$

with $2\sigma^{(1)}(\nabla\mathbf{u})=S_{ij}(\mathbf{E}(\mathbf{u}))E_{ij}(\mathbf{u})$, $2\sigma^{(2)}(\nabla^{(2)}\mathbf{u})=H_{\alpha 33}(\mathbf{G}(\mathbf{u}))G_{\alpha 33}(\mathbf{u})$ (repeated Latin and Greek indices run over $\{1,2,3\}$ and $\{1,2\}$, respectively; $\mathbf{E}$ is the standard strain and $\mathbf{G}$ the *hyperstrain*, of Cartesian components $2E_{ij}(\mathbf{u})=u_{i},_{j}+u_{j},_{i}$ and $G_{\alpha 33}(\mathbf{u})=u_{\alpha},_{33}$). The constitutive dependences are:

$$S_{\alpha\beta}=2\mu E_{\alpha\beta}+(\lambda(E_{11}+E_{22})+\tau_2E_{33})\delta_{\alpha\beta},\qquad S_{3\alpha}=2\gamma E_{3\alpha},\qquad S_{33}=\tau_1E_{33}+\tau_2(E_{11}+E_{22})$$

for the standard stress $\mathbf{S}$ on $\mathbf{E}$, and $H_{\alpha 33}=\tau_PG_{\alpha 33}$ for the *hyperstress* $\mathbf{H}$ on $\mathbf{G}$ (so that the material is transversely isotropic with respect to the axial direction $\mathbf{e}_3$). Positivity of the stored-energy density is guaranteed by the inequalities $\mu>0,\ \gamma>0,\ \tau_1>0,\ \tau_1(\lambda+\mu)-\tau_2^2>0,\ \tau_P>0$.

Let $\mathcal{C}(\varepsilon)=\mathcal{P}(\varepsilon)\times\mathcal{R}(\varepsilon)$ be a right cylinder of cross-section $\mathcal{P}(\varepsilon)$ and axis $\mathcal{R}(\varepsilon)$ of length $2h(\varepsilon)$, comprised of an elastic material with stored-energy as above. We scale the in-plane and out-of-plane coordinates x_α, x_3, the Cartesian components u_α, u_3 of the displacement field, and the *material moduli* $\lambda,\mu,\dots,\tau_P$, as follows: $x_\alpha=\varepsilon^p\bar{x}_\alpha,\ x_3=\varepsilon^q\bar{x}_3$; $u_\alpha=\varepsilon^m\bar{u}_\alpha,\ u_3=\varepsilon^n\bar{u}_3$; $\bar\lambda=\varepsilon^{-r}\lambda,\ \bar\mu=\varepsilon^{-r}\mu,\ \bar\gamma=\varepsilon^{-u}\gamma, \bar\tau_1=\varepsilon^{-v}\tau_1,\ \bar\tau_2=\varepsilon^{-z}\tau_2,\ \bar\tau_P=\varepsilon^{-i}\tau_P$. The *scaling exponents* $(m,n,p,q;r,u,v,z,i)$ are relative integers that can be chosen according to convenience, provided they satisfy the following restrictions: $q=m-n+p,\ r=2z-v$, whose physical significance is discussed in [3]. For $\hat\ell\equiv(\hat\alpha,\hat\beta_1,\hat\beta_2,\hat\gamma,\hat\imath)$ a list of *energy exponents*

$$\begin{aligned}&\hat\alpha=-m+3n+p+v,\quad \hat\beta_1=m+n+p+u,\quad \hat\beta_2=m+n+p+z,\\&\hat\gamma=3m-n+p+2z-v,\quad \hat\imath=i+2p-q,\end{aligned}\tag{1}$$

and for each $\varepsilon>0$ fixed, a formal scaling of the stored-energy functional $\Sigma(\mathbf{u},\varepsilon)=\int_{\mathcal{C}(\varepsilon)}\sigma(\nabla\mathbf{u},\nabla^{(2)}\mathbf{u})$ produces the *auxiliary energy functional*:

$$\begin{aligned}\widehat{\Sigma}(\bar{\mathbf{u}},\varepsilon;\hat\ell)=\varepsilon^{\hat\alpha}\bigg[\int_{\mathcal{C}}\frac12\bar\tau_1\overline{E}_{33}^2\bigg]+\varepsilon^{\hat\beta_1}\bigg[\int_{\mathcal{C}}2\bar\gamma\big(\overline{E}_{13}^2+\overline{E}_{23}^2\big)\bigg]+\varepsilon^{\hat\beta_2}\bigg[\int_{\mathcal{C}}\bar\tau_2\overline{E}_{33}\big(\overline{E}_{11}+\overline{E}_{22}\big)\bigg]\\+\varepsilon^{\hat\gamma}\bigg[\int_{\mathcal{C}}\frac12\big((\bar\lambda+2\bar\mu)\big(\overline{E}_{11}+\overline{E}_{22}\big)^2-4\bar\mu\big(\overline{E}_{11}\overline{E}_{22}-\overline{E}_{12}^2\big)\big)\bigg]+\varepsilon^{\hat\imath_1}\bigg[\int_{\mathcal{C}}\frac12\bar\tau_P\sum_\alpha(\bar{u}_\alpha,_{33})^2\bigg],\end{aligned}$$

with $\mathcal{C}$ the right cylinder of cross section $\mathcal{P}(1)$ and length $2h(1)$. All the integral addenda are positive, except the third; we regard as admissible any displacement field $\bar{\mathbf{u}}$ at which they all have finite value.

As shown in [3], the requirement that, at each admissible displacement, the auxiliary energy functional stay bounded above as $\varepsilon\to 0+$:

$$\lim_{\varepsilon\to 0+}\widehat{\Sigma}(\bar{\mathbf{u}},\varepsilon;\hat\ell)=:\overline{\Sigma}(\bar{\mathbf{u}};\hat\ell)<+\infty\tag{2}$$

can be satisfied for various choices of $\hat\ell$ (all consistent with the identity $\hat\alpha-2\hat\beta_2+\hat\gamma=0$), so as to produce an exhaustive *taxonomy of limit functionals* $\overline{\Sigma}(\bar{\mathbf{u}};\hat\ell)$, each of which, after inverse scaling, can be shown to rule a well-identified structure problem.

The choice of energy exponents leading to the Reissner–Mindlin plate theory is: $\hat{\alpha} < 0$, $\hat{\beta}_1 = \hat{\gamma} = 0$, $\iota < 0$, and is compatible with infinitely many lists of scaling exponents. Now, when methods of asymptotic analysis [1] or variational convergence are used, the customary (domain, displacement) scaling is: $m = 1$, $n = 0$, $p = 0$, $q = 1$; with this, (1) imply that $\hat{\alpha} = -1 + v$, $\hat{\beta}_1 = 1 + u$, $\hat{\beta}_2 = 1 + z$, $\hat{\gamma} = 3 + 2z - v$, $\hat{\iota} = i - 1$. We then take $u = -1$ and $2z - v = -3 = r$ ($r = -3$ is another customary choice), and we are left with the following limitations: $v < 1$, $i < 1$, that we satisfy by choosing $v = -1 = i$. All in all, the ε–family of auxiliary energy functionals we select is

$$\widehat{\Sigma}_{\mathrm{RM}}(\bar{\mathbf{u}}, \varepsilon) := \widehat{\Sigma}(\bar{\mathbf{u}}, \varepsilon; \hat{\ell}_{\mathrm{RM}}) = \int_{\mathcal{C}} \sigma_\varepsilon\big(\overline{\mathbf{E}}, \overline{\mathbf{G}}\big), \quad \hat{\ell}_{\mathrm{RM}} \equiv (-2, 0, -1, 0, -2), \quad \overline{\mathbf{E}} = \mathbf{E}(\bar{\mathbf{u}}), \quad \overline{\mathbf{G}} = \mathbf{G}(\bar{\mathbf{u}}), \tag{3}$$

where the integrand $\sigma_\varepsilon(\overline{\mathbf{E}}, \overline{\mathbf{G}}) := \sigma_\varepsilon^{(1)}(\overline{\mathbf{E}}) + \sigma_\varepsilon^{(2)}(\overline{\mathbf{G}})$ is positive, and where

$$\begin{aligned}\sigma_\varepsilon^{(1)}(\overline{\mathbf{E}}) &:= 2\bar{\gamma}\big(\overline{E}_{13}^2 + \overline{E}_{23}^2\big) + \frac{1}{2}(\bar{\lambda} + 2\bar{\mu})\big(\overline{E}_{11} + \overline{E}_{22}\big)^2 - 4\bar{\mu}\big(\overline{E}_{11}\overline{E}_{22} - \overline{E}_{12}^2\big) \\ &\quad + \varepsilon^{-2}\left[\frac{1}{2}\bar{\tau}_1\big(\overline{E}_{33}\big)^2\right] + \varepsilon^{-1}\big[\bar{\tau}_2\big(\overline{E}_{11} + \overline{E}_{22}\big)\overline{E}_{33}\big], \qquad \sigma_\varepsilon^{(2)}(\overline{\mathbf{G}}) := \varepsilon^{-2}\left[\frac{1}{2}\bar{\tau}_P \sum_\alpha \overline{G}_{\alpha 33}^2\right].\end{aligned} \tag{4}$$

With (3) and (4), the boundedness requirement (2) has the following consequences:

$$\begin{aligned}&\overline{\Sigma}_{\mathrm{RM}}(\bar{\mathbf{u}}) = \int_{\mathcal{C}} \left(2\bar{\gamma}\big(\overline{E}_{13}^2 + \overline{E}_{23}^2\big) + \frac{1}{2}(\bar{\lambda} + 2\bar{\mu})\big(\overline{E}_{11} + \overline{E}_{22}\big)^2 - 4\bar{\mu}\big(\overline{E}_{11}\overline{E}_{22} - \overline{E}_{12}^2\big)\right); \\ &\bar{u}_{3,3} = 0, \quad \bar{u}_{\alpha,33} = 0;\end{aligned} \tag{5}$$

the general solution of the system of PDEs $(5)_{2,3}$ is:

$$\bar{u}_\alpha = \hat{v}_\alpha(\bar{x}_1, \bar{x}_2) + \bar{x}_3\hat{\varphi}_\alpha(\bar{x}_1, \bar{x}_2), \qquad \bar{u}_3 = \hat{u}_3(\bar{x}_1, \bar{x}_2), \tag{6}$$

that is to say, the kinematical *Ansatz* the Reissner–Mindlin plate theory is based on. Note that

$$\min_{\rho \in \mathbb{R}} \sigma_\varepsilon^{(1)}\big(\overline{\mathbf{E}} + \rho\mathbf{e}_3 \otimes \mathbf{e}_3\big) = 2\bar{\gamma}\big(\overline{E}_{13}^2 + \overline{E}_{23}^2\big) + \frac{1}{2}(\tilde{\lambda} + 2\bar{\mu})\big(\overline{E}_{11} + \overline{E}_{22}\big)^2 - 4\bar{\mu}\big(\overline{E}_{11}\overline{E}_{22} - \overline{E}_{12}^2\big) =: \tilde{\sigma}\big(\overline{\mathbf{E}}\big), \tag{7}$$

where $\tilde{\lambda} := \bar{\lambda} - \bar{\tau}_2^2/\bar{\tau}_1$ and the minimum is attained at $(\overline{E}_{33})_{\min} = -\varepsilon(\bar{\tau}_2/\bar{\tau}_1)\sum_\alpha \overline{E}_{\alpha\alpha}$; note also that $\overline{\Sigma}_{\mathrm{RM}}(\bar{\mathbf{u}}) > \widetilde{\Sigma}_{\mathrm{RM}}(\bar{\mathbf{u}}) := \int_C \tilde{\sigma}(\overline{\mathbf{E}}(\bar{\mathbf{u}})) \geqslant 0$. As a candidate Γ-limit of the family $\widehat{\Sigma}_{\mathrm{RM}}(\cdot, \varepsilon)$, we pick the functional $\widetilde{\Sigma}_{\mathrm{RM}}$, defined over the class (6) of displacement fields. A detailed justification of this choice will be given elsewhere.

3. Variational convergence

Let the family of functionals $\widehat{\Sigma}_{\mathrm{RM}}(\cdot, \varepsilon)$ in (3), (4) be defined over the space of *admissible displacements*

$$\mathcal{A} := \big\{\bar{\mathbf{u}} \in H^1\big(\mathcal{C}, \mathbb{R}^3\big) \mid \bar{u}_{\alpha,33} \in L^2(\mathcal{C}),\ \bar{\mathbf{u}} = \mathbf{0} \text{ on } \partial\mathcal{P} \times \mathcal{R}\big\}$$

(for brevity and simplicity, we here limit ourselves to clamped–edge boundary conditions), and let the space of *Reissner–Mindlin displacements* be

$$\mathcal{RM} := \big\{\bar{\mathbf{u}} \in H^1\big(\mathcal{C}, \mathbb{R}^3\big) \mid \exists \hat{v}_\alpha, \hat{\varphi}_\alpha, \hat{u}_3 \in H_0^1(\mathcal{P})\colon\ \bar{u}_\alpha = \hat{v}_\alpha(\bar{x}_1, \bar{x}_2) + \bar{x}_3\hat{\varphi}_\alpha(\bar{x}_1, \bar{x}_2),\ \bar{u}_3 = \hat{u}_3(\bar{x}_1, \bar{x}_2)\big\}.$$

This displacement class, also considered in [2], is here brought to attention as a consequence of the requirement that a suitable family of functionals be bounded [3]; such a requirement is the content of the next lemma, whose easy proof we omit.

Lemma 1. *Let* $\{\bar{\mathbf{u}}^\varepsilon\} \subset \mathcal{A}$ *be any sequence such that* $\sup_{\varepsilon \in (0,1)} \widehat{\Sigma}_{\mathrm{RM}}(\bar{\mathbf{u}}^\varepsilon, \varepsilon) < +\infty$. *Then, there are a subsequence* (*not relabeled*) *of* $\{\bar{\mathbf{u}}^\varepsilon\}$ *and a displacement field* $\bar{\mathbf{u}} \in \mathcal{RM}$ *such that* $\bar{\mathbf{u}}^\varepsilon \rightharpoonup \bar{\mathbf{u}}$ *in* $H^1(\mathcal{C}, \mathbb{R}^3)$.

Theorem 2. *Let the extended Reissner–Mindlin functionals* $\widehat{\Sigma}_{\mathrm{RM}}^e(\cdot, \varepsilon)$, $\widetilde{\Sigma}^e(\cdot)$ *over* $L^2(\mathcal{C}, \mathbb{R}^3)$ *be*:

$$\widehat{\Sigma}_{\mathrm{RM}}^e(\bar{\mathbf{u}}, \varepsilon) := \begin{cases} \widehat{\Sigma}_{\mathrm{RM}}(\bar{\mathbf{u}}, \varepsilon) & \text{if } \bar{\mathbf{u}} \in \mathcal{A}, \\ +\infty & \text{otherwise}, \end{cases} \quad \text{and} \quad \widetilde{\Sigma}_{\mathrm{RM}}^e(\bar{\mathbf{u}}) := \begin{cases} \widetilde{\Sigma}_{\mathrm{RM}}(\bar{\mathbf{u}}) & \text{if } \bar{\mathbf{u}} \in \mathcal{RM}, \\ +\infty & \text{otherwise}. \end{cases}$$

Then, the family $\widehat{\Sigma}_{\mathrm{RM}}^e(\cdot, \varepsilon)$ *sequentially* Γ*-converges to* $\widetilde{\Sigma}_{\mathrm{RM}}^e$, *with respect to the* $L^2(\mathcal{C})$ *topology, i.e.,*

(i) [Liminf Inequality] *for every sequence of positive numbers* ε_k *converging to* 0, *and for every pair of a sequence* $\{\bar{\mathbf{u}}^{\varepsilon_k}\} \subset L^2(\mathcal{C}, \mathbb{R}^3)$ *and field* $\bar{\mathbf{u}} \in L^2(\mathcal{C}, \mathbb{R}^3)$ *such that* $\bar{\mathbf{u}}^{\varepsilon_k} \to \bar{\mathbf{u}}$ *in* $L^2(\mathcal{C}, \mathbb{R}^3)$,

$$\liminf_{\varepsilon_k \to 0} \widehat{\Sigma}^e_{\mathrm{RM}}(\bar{\mathbf{u}}^{\varepsilon_k}, \varepsilon_k) \geqslant \widetilde{\Sigma}^e_{\mathrm{RM}}(\bar{\mathbf{u}});$$

(ii) [Recovery Sequence] *for every sequence of positive numbers* ε_k *converging to* 0, *and for every field* $\bar{\mathbf{u}} \in L^2(\mathcal{C}, \mathbb{R}^3)$, *there is a sequence* $\{\bar{\mathbf{u}}^{\varepsilon_k}\} \subset L^2(\mathcal{C}, \mathbb{R}^3)$ *such that* $\bar{\mathbf{u}}^{\varepsilon_k} \to \bar{\mathbf{u}}$ *in* $L^2(\mathcal{C}, \mathbb{R}^3)$, *and*

$$\limsup_{\varepsilon_k \to 0} \widehat{\Sigma}^e_{\mathrm{RM}}(\bar{\mathbf{u}}^{\varepsilon_k}, \varepsilon_k) \leqslant \widetilde{\Sigma}^e_{\mathrm{RM}}(\bar{\mathbf{u}}).$$

Proof. Let $\{\bar{\mathbf{u}}^\varepsilon\} \subset L^2(\mathcal{C}, \mathbb{R}^3)$ and $\bar{\mathbf{u}} \in L^2(\mathcal{C}, \mathbb{R}^3)$ be such that $\bar{\mathbf{u}}^\varepsilon \to \bar{\mathbf{u}}$ in $L^2(\mathcal{C}, \mathbb{R}^3)$. Without loss of generality, we may assume that $\liminf_{\varepsilon \to 0} \widehat{\Sigma}^e_{\mathrm{RM}}(\bar{\mathbf{u}}^\varepsilon, \varepsilon) < +\infty$, and hence that $\sup_\varepsilon \widehat{\Sigma}^e_{\mathrm{RM}}(\bar{\mathbf{u}}^\varepsilon, \varepsilon) < +\infty$. By Lemma 1, it then follows that $\bar{\mathbf{u}}^\varepsilon \rightharpoonup \bar{\mathbf{u}}$ in $H^1(\mathcal{C}, \mathbb{R}^3)$, and that $\bar{\mathbf{u}} \in \mathcal{RM}$. We then have:

$$\liminf_{\varepsilon \to 0} \widehat{\Sigma}^e_{\mathrm{RM}}(\bar{\mathbf{u}}^\varepsilon, \varepsilon) \geqslant \liminf_{\varepsilon \to 0} \int_{\mathcal{C}} \sigma^{(1)}_\varepsilon\big(\overline{\mathbf{E}}^\varepsilon\big) \geqslant \liminf_{\varepsilon \to 0} \int_{\mathcal{C}} \tilde{\sigma}\big(\overline{\mathbf{E}}^\varepsilon\big) \geqslant \int_{\mathcal{C}} \tilde{\sigma}\big(\overline{\mathbf{E}}\big) = \widetilde{\Sigma}^e_{\mathrm{RM}}(\bar{\mathbf{u}})$$

(the last inequality of the chain holds because of the convexity of $\tilde{\sigma}$). This proves the Liminf Inequality.

To prove that the Recovery-Sequence Condition holds, it suffices to take $\bar{\mathbf{u}} \in \mathcal{RM}$ (otherwise, the inequality is trivially satisfied). Actually, we begin by taking $\bar{\mathbf{u}}$ in $\mathcal{RM} \cap C^\infty(\mathcal{C}, \mathbb{R}^3)$. There are then smooth functions $\hat{v}_\alpha, \hat{\varphi}_\alpha, \hat{u}_3 \in H^1_0(\mathcal{P})$ such as to satisfy (6). Let $\{\varepsilon_k\}$ be a sequence of positive numbers converging to 0, and let

$$\bar{u}^{\varepsilon_k}_\alpha = \hat{v}_\alpha(\bar{x}_1, \bar{x}_2) + \bar{x}_3 \hat{\varphi}_\alpha(\bar{x}_1, \bar{x}_2), \qquad \bar{u}^{\varepsilon_k}_3 = \hat{u}_3(\bar{x}_1, \bar{x}_2) - \varepsilon_k \frac{\bar{\tau}_2}{\bar{\tau}_1} \sum_\alpha \left(\bar{x}_3 \hat{v}_{\alpha,\alpha} + \frac{\bar{x}_3^2}{2} \hat{\varphi}_{\alpha,\alpha} \right).$$

Then, it is clear that $\bar{\mathbf{u}}^{\varepsilon_k} \to \bar{\mathbf{u}}$ in $L^2(\mathcal{C}, \mathbb{R}^3)$; moreover, $\overline{E}^{\varepsilon_k}_{33} = -\varepsilon_k \frac{\bar{\tau}_2}{\bar{\tau}_1} \sum_\alpha (\hat{v}_{\alpha,\alpha} + \bar{x}_3 \hat{\varphi}_{\alpha,\alpha}) = -\varepsilon_k \frac{\bar{\tau}_2}{\bar{\tau}_1} \sum_\alpha \overline{E}^{\varepsilon_k}_{\alpha\alpha}$; with this, we deduce that $\sigma^{(1)}_{\varepsilon_k}(\overline{\mathbf{E}}^{\varepsilon_k}) = \tilde{\sigma}(\overline{\mathbf{E}}^{\varepsilon_k})$. This result, the fact that $\bar{u}^{\varepsilon_k}_{\alpha,33} = 0$, and the fact that $\overline{E}^{\varepsilon_k}_{\alpha\beta} = \overline{E}_{\alpha\beta}$ and $\overline{E}^{\varepsilon_k}_{\alpha 3} = \overline{E}_{\alpha 3} - \varepsilon_k R_{\alpha 3}$, $R_{\alpha 3} := \frac{\bar{\tau}_2}{\bar{\tau}_1} \sum_\beta (\bar{x}_3 \hat{v}_{\beta,\beta\alpha} + \frac{\bar{x}_3^2}{2} \hat{\varphi}_{\beta,\beta\alpha})$, allows us to conclude that

$$\widehat{\Sigma}^e_{\mathrm{RM}}\big(\bar{\mathbf{u}}^{\varepsilon_k}, \varepsilon_k\big) = \widetilde{\Sigma}^e_{\mathrm{RM}}(\bar{\mathbf{u}}) + \varepsilon_k \sum_\alpha \int_{\mathcal{C}} 2\bar{\gamma}\big((R_{\alpha 3})^2 - 2R_{\alpha 3}\overline{E}_{\alpha 3}\big);$$

finally, by letting ε_k to zero, the R-S Condition is proven for $\bar{\mathbf{u}} \in \mathcal{RM} \cap C^\infty(\mathcal{C}, \mathbb{R}^3)$. A classical density and diagonalization argument concludes the proof. □

Minimizing the extended functional $\widehat{\Sigma}^e_{\mathrm{RM}}$ is the same as minimizing $\widehat{\Sigma}_{\mathrm{RM}}$; an analogous remark applies to the limit functionals. Theorem 2 and standard Γ-convergence arguments imply that the minimizers of $\widehat{\Sigma}_{\mathrm{RM}}(\cdot, \varepsilon)$ converge in $L^2(\mathcal{C}, \mathbb{R}^3)$ to the minimizers of $\widetilde{\Sigma}_{\mathrm{RM}}$. Convergence of minimizers in the strong $H^1(\mathcal{C}, \mathbb{R}^3)$ topology follows from Lemma 1 and the fact that stored energies are strictly convex.

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