



Partial Differential Equations/Probability Theory

Stochastic variational inequalities for elasto-plastic oscillators [☆]

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Abstract

The purpose of this Note is to show that models used in the literature for the hysteresis effect of non-linear elasto-plastic oscillators submitted to random vibrations are equivalent to stochastic variational inequalities. This powerful tool allows to study the ergodic properties of the Markov process related to the displacement. We characterize completely the invariant measure by a partially degenerate elliptic partial differential equation, with new Dirichlet coupling conditions. **To cite this article:** *A. Bensoussan, J. Turi, C. R. Acad. Sci. Paris, Ser. I 343 (2006).*

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Résumé

Inéquation variationnelle stochastique modélisant un oscillateur elasto-plastique. On montre que les modèles représentant l'effet d'hystérésis pour les oscillateurs non-linéaires elasto-plastiques sont équivalents à une inéquation variationnelle stochastique. Cette technique puissante permet d'étudier complètement les propriétés ergodiques du processus de Markov relatif au déplacement. On caractérise complètement la mesure invariante par une équation aux dérivées partielles elliptique partiellement dégénérée, avec des conditions de Dirichlet nouvelles. **Pour citer cet article :** *A. Bensoussan, J. Turi, C. R. Acad. Sci. Paris, Ser. I 343 (2006).*

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Version française abrégée

L'objet de cette Note est de montrer que les modèles de la littérature modélisant les effets d'hystérésis dans les oscillateurs elasto-plastiques soumis à des vibrations aléatoires se réduisent à des inéquations variationnelles stochastiques. Cet outil puissant permet de caractériser complètement le processus de Markov gouvernant le déplacement. On étudie notamment ses propriétés d'ergodicité. La mesure invariante est obtenue en résolvant un problème elliptique partiellement dégénéré avec des conditions de Dirichlet non locales d'un type nouveau.

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1. Introduction

The study of nonlinear oscillators excited by white noise has received considerable interest over the past several decades (see e.g., [1,3,4,6–8] and the references therein). They represent useful models for predicting the response of mechanical structures when stressed beyond the elastic limit, i.e., excursions to the plastic regime for short periods of time. The description of the process is delicate, because of the memory due to successive plastic phases. One cannot derive simple equations for the displacement. Because of the hysteretic effect, the equations are of the type

$$\ddot{x} + c_0 \dot{x} + F[x(0, t)] = \dot{w} \quad (1)$$

with initial conditions

$$x(0) = z, \quad \dot{x}(0) = y, \quad (2)$$

where $F[x(0, t)]$ is a nonlinear functional depending on the trajectory between 0 and t , and where \dot{w} represents the vibration process (a white noise). In this Note, we show that the pair $\dot{x}, F[x(0, t)]$ can be expressed as the unique solution of a simple stochastic variational inequality. The displacement can then be recovered afterwards. We then prove ergodic properties of the solution and characterize the invariant measure, which plays an essential role in the computation of probabilities of failure.

2. Model equations

We consider for $t > 0$ the stochastic variational inequality

$$\dot{y} + c_0 y + kz = \dot{w}, \quad (\dot{z} - y)(\zeta - z) \geq 0, \quad \forall |\zeta| \leq Y, \quad |z(t)| \leq Y, \quad (3)$$

with initial conditions

$$y(0) = y, \quad z(0) = z. \quad (4)$$

Comparing to (1) we have

$$y = \dot{x}, \quad kz(t) = F[x(0, t)].$$

We can reduce (3) to a deterministic variational inequality with random input, by the transformation

$$u = y - w$$

and we get

$$\dot{u} + c_0 u + kz = -c_0 w, \quad (\dot{z} - u)(\zeta - z) \geq w(\zeta - z), \quad \forall |\zeta| \leq Y, \quad |z(t)| \leq Y.$$

For each trajectory $w(\cdot)$ of the Wiener process, the deterministic variational inequality is solved by standard methods. Note that

$$\dot{z} = y \mathbb{1}_{\{-Y < z < Y\}}.$$

3. Markov properties

An application of Ito's formula and straightforward calculation yield that the white noise driven elasto-plastic oscillator satisfies the following energy estimate:

Theorem 3.1. *For $t > 0$ we have $E\{|y(t)|^2\} \leq y^2 e^{-c_0 t} + c_1$, where $c_1 = \frac{k^2 Y^2}{c_0} + 2$.*

The process $z(t), y(t)$ is a Markov process on the metric space $S = \{(z, y), -Y \leq z \leq Y, y \in R\}$ equipped with the Borel σ -algebra on S , denoted by Σ . Let $C_b(S)$ be the space of continuous and bounded functions on (S, Σ) and let $\phi \in C_b(S)$. The semi-group associated to the process $z(t), y(t)$ is defined by

$$P(t)\phi(z, y) = E\phi(z(t), y(t)).$$

Its infinitesimal generator, A , is defined for smooth functions $\phi \in S$ by

$$A\phi(z, y) = \begin{cases} \phi_z y + \phi_y(-c_0 y - kz) + \frac{1}{2}\phi_{yy}, & \text{if } -Y < z < Y, \\ \phi_y(-Y, y)(-c_0 y + kY) + \frac{1}{2}\phi_{yy}(-Y, y), & \text{if } z = -Y, y < 0, \\ \phi_y(Y, y)(-c_0 y - kY) + \frac{1}{2}\phi_{yy}(Y, y), & \text{if } z = Y, y > 0. \end{cases}$$

Thanks to the result of Theorem 3.1 and using standard arguments, it is easy to show that the process $z(t), y(t)$ has an invariant measure. We state

Theorem 3.2. *There exists an invariant probability measure on S, Σ, μ , such that $\forall t \geq 0$ and $\forall \phi \in C_b(S) \mu(P(t)\phi) = \mu(\phi)$.*

4. Ergodicity

An invariant measure μ satisfies $\mu(A\phi) = 0, \forall \phi$, smooth function on S . We are going to prove the following:

Theorem 4.1. *The invariant measure is unique, is represented by a density, $m(z, y)$, and in a weak sense the following relations hold:*

$$\begin{aligned} -y \frac{\partial m}{\partial z} + \frac{\partial}{\partial y}(m(c_0 y + kz)) + \frac{1}{2} \frac{\partial^2 m}{\partial y^2} &= 0, \quad -Y < z < Y, y \in \mathbb{R}, m(-Y, y) = 0, y \geq 0, m(Y, y) = 0, y \leq 0, \\ -m(-Y, y)y + \frac{\partial}{\partial y}(m(-Y, y)(c_0 y - kY)) + \frac{1}{2} \frac{\partial^2 m(-Y, y)}{\partial y^2} &= 0, \quad \text{if } y < 0, \\ m(Y, y)y + \frac{\partial}{\partial y}(m(Y, y)(c_0 y + kY)) + \frac{1}{2} \frac{\partial^2 m(Y, y)}{\partial y^2} &= 0, \quad \text{if } y > 0. \end{aligned} \tag{5}$$

Moreover the process $z(t), y(t)$ is ergodic.

4.1. Partial differential equation with right-hand side

We introduce some notation

$$\begin{aligned} L\eta &= \eta_z + \eta_y(-c_0 y - kz) + \frac{1}{2}\eta_{yy}, \\ B_Y \eta &= \eta_y(-c_0 y - kY) + \frac{1}{2}\eta_{yy}, \quad B_{-Y} \eta = \eta_y(-c_0 y + kY) + \frac{1}{2}\eta_{yy}. \end{aligned}$$

Let $\psi(y) = \log |y| + K$. The number K is chosen so that

$$\psi(y) > 0, \quad \text{if } |y| > y_1.$$

We consider the problem

$$\begin{aligned} Lu + f &= 0 \quad \text{in } (-Y, Y) \times \mathbb{R}, \quad B_Y u + f = 0, \quad \text{if } 0 < y, \quad B_{-Y} u + f = 0, \quad y < 0, \\ u\psi^{-1} &\text{ is bounded for } y > \bar{y} \text{ or } y < -\bar{y}. \end{aligned} \tag{6}$$

We are going to prove the following result:

Theorem 4.2. *A necessary and sufficient condition for (6) to have a solution is $v(f) = 0$, where $v(f)$ is a probability having a density*

$$v(f) = \int_{-Y}^Y \int_{-\infty}^{\infty} m(z, y) f(z, y) dz dy + \int_0^{\infty} m(Y, y) f(Y, y) dy + \int_{-\infty}^0 m(-Y, y) f(-Y, y) dy.$$

The solution u is unique up to an additive constant. The density $m(z, y)$ is a weak solution of Eq. (5).

The solution of Eq. (6) is obtained by solving a sequence of Interior, Exterior Dirichlet problems, following the approach of Khasminskii, see [5].

4.2. The interior Dirichlet problem

Let $\bar{y}_1 > 0$, and $\phi(z)$ denote a bounded Borel function and consider the interior Dirichlet problem

$$\begin{aligned} L\eta &= 0, & \text{if } -Y < z < Y, \quad -\bar{y}_1 < y < \bar{y}_1, \\ B_Y\eta &= 0, & 0 < y < \bar{y}_1; \quad B_{-Y}\eta = 0, & \quad -\bar{y}_1 < y < 0, \\ \eta(z, \bar{y}_1) &= \phi(z) & \text{and } \eta(z, -\bar{y}_1) = 0. \end{aligned} \tag{7}$$

Since $\eta(Y, \bar{y}_1) = \phi(Y)$, and $\eta(-Y, -\bar{y}_1) = 0$, we can write by solving the ordinary differential equations for η at $z = Y$, and $z = -Y$, respectively,

$$\eta(Y, y) = \eta_Y I(y, \bar{y}_1) + \phi(Y)I(0, y), \quad 0 < y \leq \bar{y}_1; \quad \eta(-Y, y) = \eta_{-Y} I(-y, \bar{y}_1), \quad -\bar{y}_1 \leq y < 0, \tag{8}$$

where η_Y , and η_{-Y} are constants, and

$$I(a, b) = \frac{\int_a^b \exp(c_0\lambda^2 + 2kY\lambda) \, d\lambda}{\int_0^{\bar{y}_1} \exp(c_0\lambda^2 + 2kY\lambda) \, d\lambda}.$$

Problem (7) is a nonstandard Dirichlet problem, since the Dirichlet data depends on the unknown function in a nonlocal way.

Uniqueness of the solution of (7) is an easy consequence of maximum principle type arguments. In order to establish existence of solutions we approximate (7) by the mixed Dirichlet–Neumann problem:

$$\begin{aligned} \frac{\epsilon}{2}\eta_{zz}^\epsilon + L\eta^\epsilon &= 0, & -Y < z < Y, \quad -\bar{y}_1 < y < \bar{y}_1, \\ \eta^\epsilon(Y, y) &= \eta_Y^\epsilon I(y, \bar{y}_1) + \phi(Y)I(0, y), & 0 < y < \bar{y}_1; \quad \eta^\epsilon(-Y, y) = \eta_{-Y}^\epsilon I(-y, \bar{y}_1), & \quad -\bar{y}_1 < y < 0, \\ \eta^\epsilon(z, \bar{y}_1) &= \phi(z), \quad \eta^\epsilon(z, -\bar{y}_1) = 0, & \eta_z^\epsilon(Y, y) = 0, & \quad -\bar{y}_1 < y < 0, \quad \eta_z^\epsilon(-Y, y) = 0, & \quad 0 < y < \bar{y}_1. \end{aligned} \tag{9}$$

The uniqueness of the solution of (9) follows from maximum principle type arguments. The existence will be proved by variational arguments. Define $D_1 = (-Y, Y) \times (-\bar{y}_1, \bar{y}_1)$ and

$$\begin{aligned} K &= \{v \in H^1(D_1): v(z, \bar{y}_1) = \phi(z), \quad v(z, -\bar{y}_1) = 0, \quad v(Y, y) = v_Y I(y, \bar{y}_1) + \phi(Y)I(0, y), \\ & \quad 0 \leq y \leq \bar{y}_1; \quad v(-Y, y) = v_{-Y} I(-y, \bar{y}_1), \quad -\bar{y}_1 \leq y \leq 0, \quad |v_Y|, |v_{-Y}| \leq \|\phi\|_{L^\infty}\}. \end{aligned} \tag{10}$$

If $v \in K$, then necessarily $v_Y = v(Y, 0)$ and $v_{-Y} = v(-Y, 0)$. Note that if $u \in K$, and $v \in H^1(D_1)$,

$$v(z, \bar{y}_1) = v(z, -\bar{y}_1) = 0, \quad v(Y, y) = 0, \quad 0 < y < \bar{y}_1, \quad v(-Y, y) = 0, \quad -\bar{y}_1 < y < 0,$$

then $u + v \in K$. We shall assume that K is not empty. We observe that if $\phi \in H^1(-Y, Y)$, then K is nonempty. In particular the function

$$v(z, y) = \phi(z) \frac{\int_{-\bar{y}_1}^y \exp(c_0\lambda^2 + 2kz\lambda) \, d\lambda}{\int_{-\bar{y}_1}^{\bar{y}_1} \exp(c_0\lambda^2 + 2kz\lambda) \, d\lambda}$$

belongs to K for $\phi \in H^1(-Y, Y)$.

We consider the variational form on $H^1(D_1)$:

$$a(u, v) = \iint_{D_1} \left[\frac{\epsilon}{2} u_z v_z + \frac{1}{2} u_y v_y - u_z y v + u_y (c_0 y + kz) v \right] dz dy.$$

The problem (9) admits the variational form

$$a(u, v - u) \geq 0, \quad \forall v \in K, \quad u \in K.$$

For α sufficiently large, the bilinear form $a(u, v) + \alpha(u, v)$ is coercive on $H^1(D_1)$. For $f \in L^2(D_1)$ we can solve the variational inequality

$$a(u, v - u) + \alpha(u, v - u) \geq (f, v - u), \quad \forall v \in K, u \in K.$$

We then define the map $u = T_\alpha w$

$$a(u, v - u) + \alpha(u, v - u) \geq \alpha(w, v - u), \quad \forall v \in K, u \in K$$

and we have the next result:

Lemma 4.3. *Assume $\|w\|_{L^\infty} \leq \|\phi\|_{L^\infty}$, then $\|u\|_{L^\infty} \leq \|\phi\|_{L^\infty}$.*

Moreover, if $\|w\|_{L^\infty} \leq \|\phi\|_{L^\infty} = \gamma$, then taking $v = u_0 \in K$ we have

$$a(u, u) + \alpha|u|^2 \leq a(u, u_0) + \alpha(u, u_0 - \alpha(w, u_0 - u)) \leq a(u, u_0) + \alpha(u, u_0) + \alpha\gamma|u_0 - u|_{L^1(D_1)}$$

which implies easily $\|u\|_{H^1(D_1)} \leq M$ for an appropriate constant M depending only on the H^1 norm of u_0 and on γ .

We consider T_α acting on $\bar{K} = \{w \in K: \|w\|_{L^\infty} \leq \|\phi\|_{L^\infty}, \|w\|_{H^1(D_1)} \leq M\}$. It is clear that T_α maps \bar{K} into itself. The set \bar{K} is a compact subset of $L^2(D_1)$, and T_α is continuous. Hence T_α has a fixed point, which proves existence.

Next we establish some integral estimates on the partial derivatives of η^ϵ . We have

$$\int \int_{D_1} (\eta_y^\epsilon)^2 dz dy \leq C; \quad \epsilon \int \int_{D_1} (\eta_z^\epsilon)^2 dz dy \leq C, \tag{11}$$

$$\int \int_{D_{1\delta}} (\eta_z^\epsilon)^2 dz dy \leq C_\delta, \quad \forall \delta > 0, \text{ where } D_{1\delta} = (-Y, Y) \times (\delta, \bar{y}_1) \cup (-Y, Y) \times (-\bar{y}_1, -\delta). \tag{12}$$

We can then extract a subsequence such that

$$\begin{aligned} \eta^\epsilon &\rightarrow \eta \quad \text{in } L^2(D_1; H^1(D_1)) \text{ weakly,} \\ \eta^\epsilon(Y, 0) &\rightarrow \eta_Y \quad \text{and} \quad \eta^\epsilon(-Y, 0) \rightarrow \eta_{-Y}, \eta_z^\epsilon \rightarrow \eta_z \quad \text{in } L^2(D_{1\delta}) \text{ weakly, } \forall \delta > 0. \end{aligned}$$

So the limit satisfies

$$\|\eta\| \leq \|\phi\|, \quad \eta_y \in L^2(D_1), \quad \eta_z \in L^2(D_{1\delta}), \quad \forall \delta > 0. \tag{13}$$

Moreover, η satisfies (7), (8) since the boundary conditions have a meaning in view of the regularity of η described in (13).

Note that if $\theta(y)$ is a smooth function which vanishes on $(-\delta, \delta)$ and near the boundaries \bar{y}_1 and $-\bar{y}_1$, then $\eta\theta$ is smooth (for z up to the boundaries $-Y, Y$).

If the assumption, that K is nonempty is removed the variational formulation is no more valid. We proceed by regularization. We consider a sequence of smooth functions $\phi^\epsilon \rightarrow \phi$ in $L^2(-Y, Y)$ and $\|\phi^\epsilon\|_{L^\infty} \leq \|\phi\|_{L^\infty}$. We associate to ϕ^ϵ the solution η^ϵ of (7). We get $\|\eta^\epsilon\|_{L^\infty}$. Let $\theta > 0$ be a smooth function with compact support in $(-\bar{y}_1, \bar{y}_1)$, and test equation (7). We can show that $\eta_y^\epsilon \theta$ remains bounded in $L^2(D_1)$; $\eta_z^\epsilon \theta$ remains bounded in $L^2(D_{1\delta})$, $\forall \delta > 0$. Moreover, $\eta_{\pm Y}^\epsilon, \phi^\epsilon(Y)$ are bounded. We can thus extract a subsequence $\eta^\epsilon \rightarrow \eta$ in L^∞ weak star, $\eta_y^\epsilon \theta \rightarrow \eta_y \theta$ in $L^2(D_1)$, $\eta_z^\epsilon \theta \rightarrow \eta_z \theta$ in $L^2(D_{1\delta})$, $\forall \delta > 0$, and $\eta^\epsilon(Y, y)\theta(y) \rightarrow \theta(y)[\eta_Y I(y, \bar{y}_1) + \phi(Y)I(0, y)]$ pointwise. It follows that $B_Y \eta = 0$ for $0 < y < \bar{y}_1$ and $B_{-Y} \eta = 0$ for $-\bar{y}_1 < y < 0$ and the equation is satisfied in D_1 in the sense of distributions.

Thus we have a solution to (7) such that for θ smooth function of y with compact support in $(-\bar{y}_1, \bar{y}_1)$, $\|\eta\|_{L^\infty} \leq \|\phi\|_{L^\infty}$, $\eta_y \theta \in L^2(D_1)$, $\eta_z \theta \in L^2(D_{1\delta})$, $\forall \delta > 0$ and $\eta\theta$ is smooth outside a neighborhood of $y = 0$.

4.3. The exterior Dirichlet problem

Let $\bar{y} > 0$ and consider the problem

$$L\eta = 0, \quad -Y < z < Y, \quad y > \bar{y}, \quad \eta(Y, y) = h(Y), \quad y > \bar{y}, \quad \eta(z, \bar{y}) = h(z). \tag{14}$$

We argue uniqueness of solutions of (14) as follows: Let $\psi = \log y + K$ such that $\psi > 0$ for $y = \bar{y}$. Assume also $\bar{y} > kY/c_0$, and define $\eta = u\psi$. Considering the equation for u , noting that $u(z, y) \rightarrow 0$ as $y \rightarrow \infty$ and using maximum principle arguments, one can check that $u = 0$.

To show existence, we consider the Cauchy problem

$$L\eta^R = 0, \quad -Y < z < Y, \quad y > \bar{y}, \quad \eta^R(z, \bar{y}) = h(z), \quad \eta^R(z, R) = 0, \quad \eta^R(Y, y) = h(Y) \left[1 - \frac{I(\bar{y}, y)}{I(\bar{y}, R)} \right].$$

We have $\|\eta^R\|_{L^\infty} \leq \|h\|_{L^\infty}$, $\eta^R(Y, y) \rightarrow h(Y)$ as $R \rightarrow \infty$. We can assume $h \geq 0$ (otherwise decompose $h = h^+ - h^-$), and then $\eta^R(z, y) \uparrow$ as $R \uparrow \infty$, and the limit satisfies (14) in the sense of distributions.

4.4. The ergodic operator

We take $\bar{y} < \bar{y}_1$. We introduce the notation $\Gamma_1 = [-Y, Y] \times \{\bar{y}_1\} \cup [-Y, Y] \times \{-\bar{y}_1\}$. A Borel function $\Phi(z, y)$ on Γ_1 is given by $\phi_+(z) = \Phi(z, \bar{y}_1)$ if $y = \bar{y}_1$ and $\phi_-(z) = \Phi(z, -\bar{y}_1)$ if $y = -\bar{y}_1$.

We first solve the interior Dirichlet problem (7), (8) on D_1 for $\zeta(z, y)$ with boundary conditions $\zeta(z, \bar{y}_1) = \phi_+(z)$ and $\zeta(z, -\bar{y}_1) = \phi_-(z)$. Note that in (8) we get an extra term in $\zeta(-Y, y)$, i.e., $\zeta(-Y, y) = \zeta_{-Y} I(-y, \bar{y}_1) + \phi_-(-Y) I(0, -y)$ for $-\bar{y}_1 < y < 0$. It follows that $\|\zeta\| \leq \|\Phi\| = \text{Max}(\|\phi_+\|, \|\phi_-\|)$ and $\zeta_y \in L^2(D_1)$. Moreover, if θ is a smooth function of y with compact support on $(\delta, \bar{y}_1) \cup (-\bar{y}_1, -\delta)$, then $\zeta\theta$ is smooth in D_1 .

We then solve the exterior Dirichlet problem (14) for η on domains $\bar{D}_u = \{(z, y): -Y < z < Y, y > \bar{y}\}$ and $\bar{D}_d = \{(z, y): -Y < z < Y, y < -\bar{y}\}$, with boundary conditions $\eta(z, \bar{y}) = \zeta(z, \bar{y})$, $\eta(Y, y) = \zeta(Y, \bar{y})$ and $\eta(z, -\bar{y}) = \zeta(z, \bar{y})$, $\eta(-Y, y) = \zeta(-Y, -\bar{y})$, respectively.

It follows that $\|\eta\|_{L^\infty} \leq \|\zeta\|_{L^\infty} \leq \|\Phi\|_{L^\infty}$ and if θ_u and θ_d are smooth functions of y with compact supports in (\bar{y}, ∞) and $(-\infty, -\bar{y})$, then $\eta\theta_u$ and $\eta\theta_d$ are smooth functions in \bar{D}_u and \bar{D}_d , respectively.

Define for $(z, y) \in \Gamma_1$ the operator P by

$$P\Phi(z, y) = \begin{cases} \eta(z, \bar{y}_1) & \text{if } y = \bar{y}_1, \\ \eta(z, -\bar{y}_1) & \text{if } y = -\bar{y}_1. \end{cases}$$

A Borel subset of Γ_1 can be written as $B = B_1 \times \{\bar{y}_1\} \cup B_2 \times \{-\bar{y}_1\}$ where B_1, B_2 are Borel subsets of $(-Y, Y)$. Namely, $B_1 = \{z: (z, \bar{y}_1) \in B\}$, $B_2 = \{z: (z, -\bar{y}_1) \in B\}$. Also $\mathbb{1}_{B_1}(z) = \mathbb{1}_B(z, \bar{y}_1)$, $\mathbb{1}_{B_2}(z) = \mathbb{1}_B(z, -\bar{y}_1)$. Take $\phi_+ = \mathbb{1}_{B_1}$, $\phi_- = \mathbb{1}_{B_2}$ in (7), (8), then

$$P\mathbb{1}_B(z, y) = \begin{cases} \zeta(z, \bar{y}_1) & \text{if } y = \bar{y}_1, \\ \zeta(z, -\bar{y}_1) & \text{if } y = -\bar{y}_1. \end{cases}$$

Let x, \tilde{x} be two points of Γ_1 . We define $\lambda_{x\tilde{x}}(B) = P\mathbb{1}_B(x) - P\mathbb{1}_B(\tilde{x})$. The operator P is ergodic if we prove that (see e.g., [2])

$$\sup \lambda_{x\tilde{x}}(B) < 1, \quad \forall x, \tilde{x} \in \Gamma \text{ and } \forall B. \tag{15}$$

Otherwise, there exists a sequence $x_k, \tilde{x}_k \in \Gamma_1$ and B_k such that $\lambda_{x_k, \tilde{x}_k}(B_k) \rightarrow 1$. If ζ_k, η_k are the solutions of the interior, exterior Dirichlet problems with $\phi_+ = \mathbb{1}_{B_{1k}}$, $\phi_- = \mathbb{1}_{B_{2k}}$, respectively, then $\zeta_k \rightarrow \zeta^*$, $\eta_k \rightarrow \eta^*$. The function ζ^* satisfies (7) and is smooth outside of neighborhoods of $y = 0$, $y = \bar{y}_1$, and $y = -\bar{y}_1$. In particular, $\zeta_k(z, \bar{y}) \rightarrow \zeta^*(z, \bar{y})$ and $\zeta_k(z, -\bar{y}) \rightarrow \zeta^*(z, -\bar{y})$ in $C^0[-Y, Y]$. Similarly η^* satisfies (14) and it is a smooth function of y outside a neighborhood of $y = \bar{y}$ and $y = -\bar{y}$. Furthermore,

$$\eta^*(z, \bar{y}) = \zeta^*(z, \bar{y}), \quad \eta^*(z, -\bar{y}) = \zeta^*(z, -\bar{y}), \quad \eta^*(Y, y) = \zeta^*(Y, \bar{y}) \quad \text{and} \quad \eta^*(-Y, y) = \zeta^*(-Y, -\bar{y}).$$

We can assert that

$$\eta_k(z, \bar{y}_1) \rightarrow \eta^*(z, \bar{y}_1) \quad \text{and} \quad \eta_k(z, -\bar{y}_1) \rightarrow \eta^*(z, -\bar{y}_1) \quad \text{in } C^0[-Y, Y]. \tag{16}$$

From the sequences x_k, \tilde{x}_k we can also extract subsequences such that $x_k \rightarrow x^*$ and $\tilde{x}_k \rightarrow \tilde{x}^*$ in Γ_1 , hence $x^* = (z^*, \bar{y}_1)$ or $(z^*, -\bar{y}_1)$ and $\tilde{x}^* = (z^*, \bar{y}_1)$ or $(z^*, -\bar{y}_1)$. Thanks to (16) we can assert that $\eta_k(x_k) \rightarrow \eta^*(x^*)$, $\eta_k(\tilde{x}_k) \rightarrow \eta^*(\tilde{x}^*)$. We must have $\eta^*(x^*) = 1$ and $\eta^*(\tilde{x}^*) = 0$.

Suppose $x^* = (z^*, \bar{y}_1)$. We cannot have $-Y < z^* < Y$ from the equation. If $z^* = -Y$, then $\eta_{z^*}^*(-Y, \bar{y}_1) < 0$, $\eta_{y\bar{y}}^*(-Y, \bar{y}_1) < 0$, which is impossible. If $z^* = Y$, then $\zeta^*(Y, \bar{y}) = 1$. However, this is impossible from the boundary condition on $\zeta^*(Y, y)$.

A similar argument holds when $x^* = (z^*, -\bar{y}_1)$. So condition (15) is satisfied.

From ergodic theory, there exists a unique invariant probability measure on Γ_1 such that

$$\left| P^n \Phi(z, \pm \bar{y}_1) - \int_{-Y}^Y \Phi(\zeta, \bar{y}_1) \pi_1(\zeta) d\zeta - \int_{-Y}^Y \Phi(\zeta, -\bar{y}_1) \pi_2(\zeta) d\zeta \right| \leq K \|\Phi\| \exp(-\rho n), \quad \rho > 0. \tag{17}$$

4.5. The interior Dirichlet problem with right-hand side

Let f be bounded in D_1 . We consider the interior Dirichlet problem

$$\begin{aligned} L\chi + f = 0 & \text{ in } D_1, & B_Y\chi + f = 0, & \quad z = Y, \quad 0 < y < \bar{y}_1, & B_{-Y}\chi + f = 0, & \quad z = -Y, \quad -\bar{y}_1 < y < 0, \\ \chi(z, \bar{y}_1) = 0 & \text{ and } & \chi(z, -\bar{y}_1) = 0. & \end{aligned} \tag{18}$$

The uniqueness of a solution of (18) is argued as the uniqueness of a solution of (7). The existence can be proven by the regularization technique used in (9). Using the test function $m_0(z, y) = \exp(-c_0(y^2 + kz^2))$ one can obtain the a priori estimates on the partial derivatives of χ similar to (11) and (12).

4.6. The exterior Dirichlet problem with right-hand side

Consider the problem

$$L\xi + f = 0, \quad -Y < z < Y, \quad y > \bar{y}, \quad B_Y\xi + f = 0, \quad z = Y, \quad y > \bar{y}, \quad \xi(z, \bar{y}) = 0. \tag{19}$$

Let $\psi(y) = \gamma(\log y + K)$, such that $\log \bar{y} + K > 0$, and $\gamma \frac{c_0}{2} \geq \|f\|$. We look for a solution of (19) such that $-\psi \leq \xi \leq \psi$. We can argue uniqueness by taking $f = 0$ and $\xi = w\psi^\alpha$ with $\alpha > 1$ and showing that $w = 0$. Existence is demonstrated by the approximation

$$L\xi^R + f = 0, \quad -Y < z < Y, \quad y > \bar{y}, \quad B_Y\xi^R + f = 0, \quad z = Y, \quad \bar{y} < y < R, \quad \xi^R(z, \bar{y}) = 0, \quad \xi^R(z, R) = 0.$$

Calling $u^R = \xi^R - \psi$ one can show that $u^R \leq 0$. The sequence ξ^R is monotone increasing and converges towards a solution.

4.7. The operator T

Let $f \in L^\infty((-Y, Y) \times R)$. We first solve (18), then (19), and then we set

$$Tf(z, y) = \left\{ \xi(z, \bar{y}_1) \text{ if } y = \bar{y}_1 \text{ and } \xi(z, -\bar{y}_1) \text{ if } y = -\bar{y}_1 \right\}.$$

This defines a linear operator from $L^\infty((-Y, Y) \times R)$ into $L^\infty(\Gamma_1)$. We define

$$\begin{aligned} \nu(f) = & \left(\int_{-Y}^Y Tf(s, \bar{y}_1) \pi_1(s) ds + \int_{-Y}^Y Tf(s, -\bar{y}_1) \pi_2(s) ds \right) \\ & \times \left(\int_{-Y}^Y T\mathbb{1}(s, \bar{y}_1) \pi_1(s) ds + \int_{-Y}^Y T\mathbb{1}(s, -\bar{y}_1) \pi_2(s) ds \right)^{-1} \end{aligned} \tag{20}$$

and the denominator is > 0 , so $\nu(f)$ is well defined. Going back to Eq. (6) we can complete the proof of Theorem 4.2, and check that $\nu(f)$ has the density m . Let ϕ be a smooth function on $[-Y, Y] \times R$ with compact support. If we take $f = -L\phi$ in $(-Y, Y) \times R$, $f(Y, y) = -B_Y\phi$, for $y > 0$, and $f(-Y, y) = -B_{-Y}\phi$, for $y < 0$, then ϕ is a solution of (6) for this f . Therefore we can assert that the representation for ν involving m with f given as above yields (5).

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