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Partial Differential Equations

Heat kernels for non-divergence operators of Hörmander type

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Abstract

We prove the existence of a fundamental solution for a class of Hörmander heat-type operators. For this fundamental solution and its derivatives we obtain sharp Gaussian bounds that allow to prove an invariant Harnack inequality. ***To cite this article: M. Bramanti et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Noyaux de la chaleur pour des opérateurs de Hörmander qui ne sont pas sous forme de divergence. Nous démontrons l'existence d'une solution fondamentale pour une classe d'opérateurs de Hörmander de type chaleur. Pour cette solution fondamentale et ses dérivées nous obtenons des bornes Gaussiennes optimales qui nous permettent de démontrer une inégalité de Harnack invariante. ***Pour citer cet article : M. Bramanti et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

1. Introduction and main results

In geometric theory of several complex variables, fully nonlinear second order equations appear, whose linearizations are nonvariational operators of Hörmander type. (See, for instance, [10], and references therein.) These kinds of operators, also arising in many other theoretical and applied settings, have the following form, in the stationary or evolutionary case:

$$L_A = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j, \tag{1}$$

$$H_A = \partial_t - \sum_{i,j=1}^{q} a_{ij}(t,x) X_i X_j \tag{2}$$

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where $X_1, X_2, \dots, X_q$ is a system of real smooth vector fields defined in some bounded domain $\Omega \subset \mathbb{R}^n$ and satisfying Hörmander's rank condition at any point. The matrix $A = \{a_{ij}(t,x)\}_{i,j=1}^q$ is real symmetric and uniformly positive definite, that is:

$$\lambda^{-1}|\xi|^2 \leqslant \sum_{i,j=1}^q a_{ij}(t,x)\xi_i\xi_j \leqslant \lambda|\xi|^2 \tag{3}$$

for some $\lambda > 0$, every $\xi \in \mathbb{R}^q$, $(t,x) \in \mathbb{R} \times \Omega$. On the coefficients a_{ij} we will make a natural assumption of Hölder continuity, expressed in terms of the distance induced by the vector fields. More precisely, if $d(x,y)$ denotes the Carnot–Carathéodory metric generated in $\mathbb{R}^n$ by the X_i's and

$$d_P\big((t,x),(s,y)\big) = \big(d(x,y)^2 + |t-s|\big)^{1/2}$$

is its 'parabolic' counterpart in $\mathbb{R}^{n+1}$, we will assume that the a_{ij}'s are Hölder continuous on $\mathbb{R} \times \Omega$ with respect to the distance d_P.

One of the main motivations of the present Note is to provide a linear framework for the aforementioned fully nonlinear equations, by performing a deep analysis of the general class of Hörmander heat-type operators (2).

Our main results, proved under the above assumptions, can be briefly summarized as follows. We shall prove the existence and basic properties of a fundamental solution h_A for the operator H_A, including representation formulas for the solution to the Cauchy problem. Also, we will show that h_A satisfies a $C^{2,\alpha}_{\mathrm{loc}}$-regularity estimate, far off the pole, which means that h_A and its derivatives $X_i h_A$, $X_i X_j h_A$, $\partial_t h_A$ are locally d_P-Hölder continuous. These results are contained in Theorem 1.1 below. Strictly related to the proof of existence of h_A, and of independent interest, are several sharp Gaussian bounds for h_A that we will establish (see Theorem 1.2). A remarkable consequence of these bounds is an invariant Harnack inequality for H_A (see Theorem 1.3).

Before going on, a clarification is in order. The operator H_A is initially assumed defined only on a cylinder $\mathbb{R} \times \Omega$ for some bounded Ω. We will extend H_A to the whole space $\mathbb{R}^{n+1}$, in such a way that, outside a compact spatial set, it coincides with the classical heat operator. Henceforth all our statements will be referred to this extended operator.

Theorem 1.1 *(Existence of a fundamental solution). Under the above assumptions, there exists a global fundamental solution $h_A(t,x;\tau,\xi)$ for H_A in $\mathbb{R}^{n+1}$, with the properties listed below.*

(i) *h_A is a continuous function away from the diagonal of $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$; $h_A(t,x;\tau,\xi) = 0$ for $t \leqslant \tau$. Moreover, for every fixed $\zeta \in \mathbb{R}^{n+1}$, $h_A(\cdot;\zeta) \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^{n+1} \setminus \{\zeta\})$, and we have*

$$H_A\big(h_A(\cdot;\zeta)\big) = 0 \quad \text{in } \mathbb{R}^{n+1} \setminus \{\zeta\}.$$

(ii) *For every $\psi \in C_0^\infty(\mathbb{R}^{n+1})$, the function $w(z) = \int_{\mathbb{R}^{n+1}} h_A(z;\zeta)\psi(\zeta)\,\mathrm{d}\zeta$ belongs to the class $C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^{n+1})$, and we have*

$$H_A w = \psi \quad \text{in } \mathbb{R}^{n+1}.$$

(iii) *Let $\mu \geqslant 0$ and $T_2 > T_1$ be such that $(T_2 - T_1)\mu$ is small enough. Then, for every $f \in C^\beta([T_1,T_2] \times \mathbb{R}^n)$ (where $0 < \beta \leqslant \alpha$) and $g \in C(\mathbb{R}^n)$ satisfying the growth condition $|f(x,t)|, |g(x)| \leqslant c\exp(\mu d(x,0)^2)$ for some constant $c > 0$, the function*

$$u(x,t) = \int_{\mathbb{R}^n} h_A(t,x;T_1,\xi)g(\xi)\,\mathrm{d}\xi + \int_{[T_1,t]\times\mathbb{R}^n} h_A(t,x;\tau,\xi)f(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi, \quad x \in \mathbb{R}^n,\ t \in (T_1,T_2],$$

belongs to the class $C^{2,\beta}_{\mathrm{loc}}((T_1,T_2) \times \mathbb{R}^n) \cap C([T_1,T_2] \times \mathbb{R}^n)$. (We explicitly remark that, under our assumptions, $d(x,0)$ is equivalent to the Euclidean norm $|x|$, for $|x| > 1$.) Moreover, u is a solution to the following Cauchy problem

$$H_A u = f \quad \text{in } (T_1,T_2) \times \mathbb{R}^n, \qquad u(\cdot,T_1) = g \quad \text{in } \mathbb{R}^n.$$

Theorem 1.2 *(Gaussian bounds). There exists a positive constant M and, for every $T > 0$, there exists a positive constant $c = c(T)$ such that, for $0 < t - \tau \leqslant T$, $x, \xi \in \mathbb{R}^n$, the following estimates hold*

$$c^{-1}\left|B\left(x,\sqrt{t-\tau}\right)\right|^{-1}\mathrm{e}^{-Md(x,\xi)^2/(t-\tau)} \leqslant h_A(t,x;\tau,\xi) \leqslant c\left|B\left(x,\sqrt{t-\tau}\right)\right|^{-1}\mathrm{e}^{-d(x,\xi)^2/M(t-\tau)},$$

$$\left|X_i h_A(t,\cdot;\tau,\xi)(x)\right| \leqslant c(t-\tau)^{-1/2}\left|B\left(x,\sqrt{t-\tau}\right)\right|^{-1}\mathrm{e}^{-d(x,\xi)^2/M(t-\tau)},$$

$$\left|X_i X_j h_A(t,\cdot;\tau,\xi)(x)\right| + \left|\partial_t h_A(\cdot,x;\tau,\xi)(t)\right| \leqslant c(t-\tau)^{-1}\left|B\left(x,\sqrt{t-\tau}\right)\right|^{-1}\mathrm{e}^{-d(x,\xi)^2/M(t-\tau)}$$

where $|B(x,r)|$ denotes the Lebesgue measure of the d-Carnot–Carathéodory ball in $\mathbb{R}^n$.

Theorem 1.3 *(Invariant Harnack inequality). Let $R_0 > 0$, $0 < h_1 < h_2 < 1$ and $\gamma \in (0,1)$. There exists a positive constant $M = M(h_1, h_2, \gamma, R_0)$ such that, for every $(\tau_0, \xi_0) \in \mathbb{R}^{n+1}$, $R \in (0, R_0]$ and for every*

$$u \in C_X^2\left(\left(\tau_0 - R^2, \tau_0\right) \times B(\xi_0, R)\right) \cap C\left(\left[\tau_0 - R^2, \tau_0\right] \times \overline{B(\xi_0, R)}\right)$$

satisfying $H_A u = 0$, $u \geqslant 0$ in $(\tau_0 - R^2, \tau_0) \times B(\xi_0, R)$, we have

$$\max\left\{u(t,x) \mid \tau_0 - h_2 R^2 \leqslant t \leqslant \tau_0 - h_1 R^2,\ x \in \overline{B(\xi_0, \gamma R)}\right\} \leqslant M u(\tau_0, \xi_0).$$

Here C_X^2 stands for the space of continuous functions u having continuous derivatives $X_i u$, $X_i X_j u$, $\partial_t u$.

All the constants in the above theorems depend on the coefficients a_{ij} only through their Hölder moduli of continuity and the constant λ in (3). Also, we note that all the results stated in Theorems 1.1, 1.2 and 1.3 hold also for the operator with lower order terms:

$$H = \partial_t - \sum_{i,j=1}^{q} a_{ij}(t,x) X_i X_j - \sum_{k=1}^{q} b_k(t,x) X_k - c(t,x)$$

(in the proof of the Harnack inequality we shall actually restrict to the case $c = 0$), provided also b_k and c are Hölder continuous and bounded.

We would like to close this section with some comments on the stationary case. A fundamental solution for the operator L_A could be obtained by integrating in the time variable the relevant heat kernel h_A, provided suitable long-time estimates of h_A are established. However, under our assumptions, proving these long-time estimates seems to be a nontrivial task. On the other hand, an invariant Harnack inequality for the stationary operators (1) obviously follows from Theorem 1.3.

2. Previous results and bibliographic remarks

Initially motivated by the classical result of Aronson about parabolic operators, the seek of Gaussian bounds for fundamental solutions of heat-type operators built with Hörmander's vector fields has a long history. For operators of the kind $H = \partial_t - \sum_{i=1}^{q} X_i^2$ with left invariant homogeneous vector fields on a Carnot group in $\mathbb{R}^n$, Gaussian bounds have been proved by Varopoulos (see [12] and references therein). In absence of a group structure, Gaussian bounds have been proved, on a compact manifold and for finite time, by Jerison–Sanchez–Calle [8], with an analytic approach and, on the whole $\mathbb{R}^{n+1}$, by Kusuoka–Stroock, (see [9] and references therein), using the Malliavin stochastic calculus.

For the operators (2) with X_i left invariant homogeneous Hörmander's vector fields on a stratified Lie group and under the above assumptions, it has been proved by Bonfiglioli and two of us [1,2] that the fundamental solution h_A exists and satisfies Gaussian bounds. An application of Gaussian estimates is the proof of an invariant Harnack inequality for the operator H_A, see [3].

3. Strategy of the proofs

Following the general strategy used in the case of homogeneous groups in [1,2], our study proceeds in two steps. First we shall consider operators of kind (2) with constant coefficients a_{ij}. For these operators, existence and basic properties of the fundamental solution are guaranteed by known results. Here the point is to prove sharp Gaussian bounds on h_A of the kind stated in Theorem 1.2 (but with bounds on derivatives of *any* order), which have to be *uniform* in the ellipticity class of the matrix A. To prove these uniform bounds, we have followed as close as possible the techniques of [8], the main new difficulties being the following: first, we have to take into account the dependence on the matrix A, getting estimates depending on A only through the number λ; second, our estimates have to be global in space, while in [8] the authors work on a compact manifold; third, we need estimates on the difference of the fundamental solutions of two operators which have no analogue in [8]. The procedure is technically involved and a crucial role is played by the Rothschild–Stein lifting theorem [11].

The second step consists in studying operators with variable Hölder continuous coefficients a_{ij}, and applying the results about constant coefficient operators to establish existence and Gaussian bounds for the fundamental solution of those operators. This is accomplished by a suitable adaptation of the classical Levi's parametric method. Regularity of $C^{2,\alpha}_{\mathrm{loc}}$-type for the fundamental solution is then proved, applying the Schauder estimates for operators of kind (2), recently proved by two of us (see [5]). Finally, thanks to Theorems 1.1 and 1.2, the proof of Harnack inequality for H_A can follow the lines drawn in [3]. The main tool here is a suitable adaptation of Krylov and Safanov's methods, already used by Fabes and Stroock [7] in the classical parabolic case, and by Kusuoka and Stroock [9] for the operators in (2) with coefficients $a_{ij} = \delta_{ij}$. However, we do not use any of the probabilistic techniques and results used in [9]. Our approach is based on the existence of the relevant Green functions on a suitable family of cylindrical open sets, which are regular for the parabolic boundary value problems. We also use a noninvariant Harnack inequality for Hörmander operators proved by Bony in [4].

All the proofs of the results announced above will appear in [6].

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