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Differential Geometry

Prescribing the scalar curvature on three dimensional spheres

Wael Abdelhedi

Département de mathématiques, faculté des sciences de Sfax, route Soukra, 3018 Sfax, Tunisia

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Abstract

By topological arguments, we set sufficient hypotheses for a given function K, on the unit sphere (S^3, g), to be the scalar curvature of a metric conformal to g. ***To cite this article: W. Abdelhedi, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Courbure scalaire prescrite sur la sphère de dimension trois. Par des arguments topologiques, on met en évidence des hypothèses suffisantes pour qu'une fonction K, donnée sur la sphère (S^3, g), soit la courbure scalaire d'une métrique conforme à g. ***Pour citer cet article : W. Abdelhedi, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

1. Introduction and the main results

Let (S^3, g) be the standard 3-sphere equipped with the standard metric. Let K be a C^2 positive function on S^3. We study the problem:

$$\begin{cases} -8\Delta_g u + 6u = K(x)u^5, \\ u > 0 \quad \text{on } S^3. \end{cases} \tag{1}$$

Under some conditions on K, we prove that this equation has at least one solution.

In this paper, we give a contribution in the spirit of Aubin and Bahri [1] and Bahri and Coron [3], using topology and Bahri's theory of critical points at infinity (see [2]). The first result here (Theorem 1.1) is that under one qualitative assumption on some of the critical points of K (assumption (C_1)) and one topological assumption on the remaining critical points of K (assumption (C_2)), then there is a positive solution of (1). This result generalizes, in particular, a result of Bahri and Coron [3] where topological contractibility assumptions on all the critical points of K are assumed (see Corollary 1.2). In Remark 1.5, we describe a situation in which Theorem 1.1 applies, but not Bahri–Coron's.

In order to state our results, we need to fix some notations and assumptions that we are using.

E-mail address: Wael_hed@yahoo.fr (W. Abdelhedi).

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Throughout this Note, we assume that K has only non-degenerate critical points $y_0, y_1, \dots, y_h$ such that $\Delta K(y_i) \neq 0$ for each $i = 0, \dots, h$ and $K(y_0) \geqslant K(y_1) \geqslant \dots \geqslant K(y_h)$. Each y_i is assumed to be of index $\operatorname{ind}(K, y_i) = 3 - k_i$. Let $I^+ = \{y_i \mid -\Delta K(y_i) > 0\}$.

Let Z be a pseudo-gradient of K of Morse–Smale type (that is the intersection of stable and unstable manifolds of the critical points of K are transverse). We assume that

$$W_s(y_i) \cap W_u(y_j) = \emptyset, \quad \text{for each } y_i \in I^+ \text{ and } y_j \notin I^+,$$

where $W_s(y_i)$ is the stable manifold of y_i and $W_u(y_j)$ is the unstable manifold of y_j for Z. For each $0 \leqslant \ell \leqslant h$, we define $X_\ell = \bigcup_{0 \leqslant j \leqslant \ell,\ y_j \in I^+} \overline{W}_s(y_j)$. We then have:

Theorem 1.1. *Assume that there exist $\ell \in \{0, \dots, h\}$ satisfying the following conditions*:

(C_1) $K(y_j)^{-1/2} > K(y_0)^{-1/2} + K(y_\ell)^{-1/2}$ *for* $j \in \{\ell + 1, \dots, h\}$ *and* $y_j \in I^+$.
(C_2) X_ℓ *is not contractible. We denote by m the dimension of the first non trivial reduced homology group.*

Then problem (1) *admits a solution.*

Corollary 1.2. *If* $\sum_{y_i \in I^+} (-1)^{3 - \operatorname{ind}(k, y_i)} \neq 1$, *then* (1) *has a solution.*

Corollary 1.3. *The solution obtained in Theorem* 1.1 *has an augmented Morse index* $\geqslant m$.

To state our next result, we need to introduce the following assumptions:

(C_3) There exist $F^+ \subset I^+$ such that $X = \bigcup_{y_i \in F^+} \overline{W}_s(y_i)$ is a stratified set in dimension $k \geqslant 1$ without boundary (in the topological sense, i.e. $X \in \mathcal{S}_k(S^3)$, the group of chains of dimension k and $\partial X = 0$).
(C_4) For all $y \in I^+ \setminus F^+$ we have $\operatorname{ind}(K, y_j) \notin \{3 - k, 3 - (k + 1)\}$.

We then have the following:

Theorem 1.4. *Under the assumptions* (C_3) *and* (C_4), *the problem* (1) *admits a solution.*

Remark 1.5. Here, we give a situation where the result of Corollary 1.2 does not give a solution to problem (1) but by Theorem 1.1 or Theorem 1.4, we derive that problem (1) admits a solution.

For this, let $K : S^3 \to \mathbb{R}$ be a function such that $I^+ = \{y_0, y_1, y_2\}$ with, $K(y_0) \geqslant K(y_1) \geqslant K(y_2)$, $\operatorname{ind}(K, y_0) = 3$, $\operatorname{ind}(K, y_1) \neq \operatorname{ind}(K, y_2) \in \{1, 2\}$ and $K(y) < K(y_1)$ for any critical point y of K which is not in I^+. It is easy to see that

$$\sum_{y_j \in I^+} (-1)^{3 - \operatorname{ind}(K, y_j)} = 1.$$

From another part, $X_1 = \overline{W}_s(y_1) = W_s(y_1) \cup \{y_0\}$ is a stratified set in dimension $\geqslant 1$, without boundary. Thus, X_1 is not contractible. We distinguish two cases:
case 1: If $K(y_2)^{-1/2} > K(y_0)^{-1/2} + K(y_1)^{-1/2}$, we deduce from Theorem 1.1 that problem (1) has a solution.
case 2: If $i(y_1) = 1$ and $i(y_2) = 2$, by Theorem 1.4 we derive that (1) has a solution.

2. Proofs of results

Problem (1) is equivalent to finding the critical points of the following function:

$$J(u) = \frac{1}{(\int_{S^3} K(x) u^6 \, \mathrm{d}v_g)^{1/3}}, \quad u \in \Sigma^+,$$

where $\Sigma^+ = \{u \in \Sigma,\ u \geqslant 0\}$ and $\Sigma = \{u \in H^1(S^3),\ |u|^2_{H^1} = 1\}$. For $a \in S^3$, $\lambda > 0$, let:

$$\delta_{(a,\lambda)}(x) = c_0\left(\frac{\lambda}{(\lambda^2+1) + (\lambda^2-1)\cos d(a,x)}\right)^{1/2},$$

where $\delta_{(a,\lambda)}(x)$ is a solution of the Yamabe problem on S^3.

Proposition 2.1. *(See Lemma 7 of* [3]*) Assume that J has no critical points in Σ^+, then the only critical points at infinity for J are $\delta(y_i,\infty)$ such that $y_i \in I^+$, where*

$$I^+ = \{y \in S^3 \mid \nabla K(y) = 0 \text{ and } -\Delta K(y) > 0\}.$$

The level of such critical point at infinity is $S^{2/3}K(y_i)^{-1/3}$, where $S = \int_{S^3}\delta^6\,\mathrm{d}v$.

Moreover, the Morse index of the critical point at infinity $\delta(y_i,\infty)$ is given by $i(y_i)_\infty = 3 - \operatorname{ind}(K, y_i)$.

Proof of Theorem 1.1. Arguing by contradiction, we assume that J has no critical points in Σ^+. Let $C_\infty(y_0, y_\ell) = S^{2/3}(\frac{1}{K(y_0)^{1/2}} + \frac{1}{K(y_\ell)^{1/2}})^{2/3}$. It follows from the result of Proposition 2.1 and the assumption (C_1) of Theorem 1.1 that, the only critical points at infinity of J under the level $c_1 = C_\infty(y_0, y_\ell) + \varepsilon$, for ε small enough, are $\delta(y_j,\infty)$ where $y_j \in I^+$ and $j \in \{0,\ldots,\ell\}$. In the neighborhood of such critical points at infinity, we have:

$$J(\alpha\delta_{(a,\lambda)} + v) = \frac{S^{2/3}}{K(a)^{1/3}}\left(1 - \frac{\Delta K(y_j)}{\lambda^2}\right) + |V|^2 \quad \text{(see [1] p. 534).}$$

In order to define our deformations, we can work as if V was zero. The deformation will extend with the same properties, to a neighborhood of zero in the V part. Thus, the unstable manifolds at infinity for the vector field $(-\partial J)$ of such critical points at infinity $W_u(y_j)_\infty$ can be described as the product of $W_s(y_j)$ (for a decreasing pseudo-gradient of K) by $[A,\infty[$ domain of the variable λ, for some positive number A large enough (see [1] p. 535). Since J has no critical points, the set $J_{c_1} = \{u \in \Sigma^+ \mid J(u) \leqslant c_1\}$ retract by deformation onto $(X_\ell)_\infty = \bigcup_{0 \leqslant j \leqslant \ell,\ y_j \in I^+} \overline{W}_u(y_j)_\infty$ which can be parameterized by $X_\ell \times [A,\infty[$.

Observe that by assumption (C_2) of Theorem 1.1 $(X_\ell)_\infty$ is not a contractible set. Now, we prove that $(X_\ell)_\infty$ is contractible in J_{c_1}. Indeed, let:

$$f : [0,1] \times (X_\ell)_\infty \longrightarrow \Sigma^+$$
$$(t,x,\lambda) \longmapsto \frac{t\delta_{(y_0,\lambda)} + (1-t)\delta_{(x,\lambda)}}{|t\delta_{(y_0,\lambda)} + (1-t)\delta_{(x,\lambda)}|_{H^1}}.$$

For $t = 0$, $f(0,x,\lambda) = \frac{1}{S}\delta_{(x,\lambda)} \in X_\infty$, f is continuous and $f(1,x,\lambda) = \frac{1}{S}\delta_{(y_0,\lambda)}$. Let $a_1, a_2 \in S^3$, $\alpha_1, \alpha_2 > 0$ and λ large enough. For $u = \alpha_1\delta_{(a_1,\lambda)} + \alpha_2\delta_{(a_2,\lambda)}$, we have:

$$J\left(\frac{u}{|u|_{H^1}}\right) \leqslant \left(S\left(\frac{1}{K(a_1)^{1/2}} + \frac{1}{K(a_2)^{1/2}}\right)\right)^{2/3}\big(1 + \mathrm{o}(1)\big),$$

where $\mathrm{o}(1) \to 0$ when $\lambda \to +\infty$ independently of t and x. Hence,

$$J\big(f(t,x,\lambda)\big) \leqslant \left(S\left(\frac{1}{K(y_0)^{1/2}} + \frac{1}{K(x)^{1/2}}\right)\right)^{2/3}\big(1 + \mathrm{o}(1)\big).$$

We claim that $K(x) \geqslant K(y_\ell)$ for any $x \in X_\ell$. Indeed, for each $x \in W_s(y_j)$ we have $\eta(s,x) \to y_j$ when $s \to +\infty$, where $\eta(s,x)$ is the decreasing flow of Z. Thus, $K(x) \geqslant K(y_j)$. Furthermore, for $0 \leqslant j \leqslant \ell$, we have $K(y_j) \geqslant K(y_\ell)$ (since $K(y_0) \geqslant K(y_1) \geqslant \cdots \geqslant K(y_h)$). Hence, our claim follows and we derive that $J(f(t,x,\lambda)) < c_1$ for any $(t,x,\lambda) \in [0,1] \times X_\ell \times [A,\infty[$.

Thus, the contraction f is performed under the level c_1. We deduce that, $(X_\ell)_\infty$ is contractible in J_{c_1}, which retracts by deformation on $(X_\ell)_\infty$, therefore $(X_\ell)_\infty$ is contractible leading to the contractibility of X_ℓ which is a contradiction. The proof of Theorem 1.1 is thereby completed. □

Proof of Corollary 1.2. We recall that K has only non degenerate critical points $y_0, y_1, \ldots, y_h$ such that $K(y_0) \geqslant K(y_1) \geqslant \cdots \geqslant K(y_h)$. For $\ell = h$, we have $X_h = \bigcup_{y_j \in I^+} \overline{W}_s(y_j)$ then $\chi(X_h) = \sum_{y_j \in I^+}(-1)^{3-\operatorname{ind} b(k,y_i)}$, where

$\chi(X_h)$ is the Euler–Poincaré characteristic of X_h (recall that for a stratified set M in dimension l, the Euler–Poincaré characterization of M is given by $\chi(M) = \sum_i (-1)^i \dim H_i(M)$, where $H_i(M)$ is the homology group in dimension i associated to M). Under the assumption of Corollary 1.2, we derive that X_h is not contractible. Hence, the result follows from Theorem 1.1. □

Proof of Theorem 1.4. Let $X = \bigcup_{y_i \in F^+} \overline{W_s}(y_i)$. By the assumption ($C_3$) of Theorem 1.4, X is a stratified set in dimension $k \geqslant 1$ without boundary. For λ large enough, we define the following set, $C_\lambda(y_0, X) = \{\alpha\delta_{y_0,\lambda} + (1-\alpha)\delta_{x,\lambda}$, $x \in X$ and $\alpha \in [0, 1]\}$.

$C_\lambda(y_0, X)$ is a contractible manifold in dimension $k+1$, that is its singularities arise in dimension $k-1$ and lower. Let $X_\infty = \bigcup_{y_i \in F^+} \overline{W_u}(y_i)_\infty$.

We argue by contradiction, we suppose that J has no critical points in Σ^+. Thus, $C_\lambda(y_0, X)$ retract by deformation on $\bigcup_{y \in H} \overline{W}_u(y)_\infty$, where $H = \{y \in I^+ \mid C_\lambda(y_0, X) \cap W_s(y)_\infty \neq \emptyset\}$.

Since $C_\lambda(y_0, X)$ is a manifold in dimension $k+1$, this manifold can be assumed to avoid the unstable manifold of every critical point at infinity $\delta_{(y,\infty)}$ of Morse index $> k+1$, i.e., $\mathrm{ind}(K, y) < 3 - (k+1)$. Thus, $H \subset \{y \in F^+ \mid \mathrm{ind}(K, y) \geqslant 3 - k\}$. More precisely, $C_\lambda(y_0, X)$ retract by deformation on $X_\infty \cup D_\infty$, where $D_\infty = \bigcup_{y \in D} W_u(y)_\infty$ and $D = \{y \in H \setminus F^+\}$.

Using the assumption (C_4) of Theorem 1.4, we derive that $\mathrm{ind}(K, y) > 3-k$ for each $y \in D$. Thus, the Morse index at infinity of the critical point at infinity $\tilde{\delta}_{(y,\infty)}$, $y \in D$ is $\leqslant k-1$, and therefore D_∞ is a stratified set of dimension at most $k-1$. Since $C_\lambda(y_0, X)$ is a contractible set, then $H_k(X_\infty \cup D_\infty) = 0$ for all $* \in \mathbb{N}^*$. Using the exact homology sequence of $(X_\infty \cup D_\infty, X_\infty)$, we have:

$$\cdots \longrightarrow H_{k+1}(X_\infty \cup D_\infty) \longrightarrow H_{k+1}(X_\infty \cup D_\infty, X_\infty) \longrightarrow H_k(X_\infty) \longrightarrow H_k(X_\infty \cup D_\infty) \longrightarrow \cdots.$$

Since $H_*(X_\infty \cup D_\infty) = 0$ for all $* \in \mathbb{N}^*$, then $H_k(X_\infty) = H_{k+1}(X_\infty \cup D_\infty, X_\infty)$.

In addition, $(X_\infty \cup D_\infty, X_\infty)$ is a stratified set of dimension at most k, so $H_{k+1}(X_\infty \cup D_\infty, X_\infty) = 0$. Thus, $H_k(X_\infty) = 0$ and therefore $H_k(X) = 0$ which is in contradiction to the assumption (C_4) of the theorem. Hence our result follows. □

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