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Exponential asymptotics and adiabatic invariance of a simple oscillator

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Abstract

An alternative proof is provided for Littlewood's asymptotic expression arising from Lorentz's problem (1911) on the adiabatic invariance of a simple pendulum. Our approach is based on a standard WKB approximation. Our proof is simpler than those of both Littlewood (1963) and Wasow (1973). If the coefficient function in their differential equation is analytic, then Littlewood's asymptotic expression can even be replaced by an exponentially small term. *To cite this article: C.H. Ou, R. Wong, C. R. Acad. Sci. Paris, Ser. I 343 (2006).*

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Résumé

Asymptotiques exponentielles et invariance adiabatique d'un oscillateur simple. On donne une autre démonstration de l'expression asymptotique que Littlewood a obtenue pour le problème de Lorentz (1911) sur l'invariance adiabatique d'un pendule simple. Notre approche repose sur l'approximation WKB habituelle. Notre démonstration est plus simple que celle de Littlewood (1963) et celle de Wasow (1973). Si le coefficient de l'équation différentielle qu'ils considèrent est analytique, alors l'expression asymptotique de Littlewood peut même être remplacée par un terme exponentiellement petit. *Pour citer cet article : C.H. Ou, R. Wong, C. R. Acad. Sci. Paris, Ser. I 343 (2006).*

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Au premier Congrès Solvay de 1911, Lorentz posa la question suivante : Quel est le comportement d'un pendule simple quand la longueur du fil diminue progressivement ? Cette question était liée à la théorie quantique de l'époque.

La formulation mathématique de ce problème est la suivante. On considère l'équation (1) (les numéros renvoient aux formules de la version anglaise), où la fonction ω vérifie les hypothèses (2).

La fonction H étant définie comme en (3), il avait été d'abord conjecturé que la relation (4) était vraie pour tout n . Cette conjecture fut effectivement établie par Littlewood [1] en 1963, mais avec une démonstration extrêmement compliquée ; voir [3, p. 214, ligne 7].

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Le même problème fut ensuite étudié par Meyer [2] et Wasow [5], en 1973. La fonction ϕ étant définie par la relation (15), l'équation (1) devient (6) avec la nouvelle variable de temps $t = \varepsilon\tau$ (un point désigne une dérivation par rapport à t ; c'est-à-dire, $\dot{u} = du/dt$). La fonction r^2 étant définie comme en (7) en fonction de t et ε , Wasow montre alors que la relation (8) est vraie pour tout $n \geq 0$ sous des hypothèses semblables à (2), ce qui équivaut évidemment à la relation (4) de Littlewood. L'approche de Wasow consiste d'alord à remplacer (5) par un système d'équations du premier ordre, puis à travailler sur une équation de Riccati obtenue après des transformations matricielles.

Dans cette Note, nous montrons que, après une application de la transformation de Liouville, la relation (8) peut être simplement établie par l'approximation WKB bien connue. Nous montrons aussi que, si la fonction ϕ est analytique en la variable complexe t avec un zéro ou une branche hors de l'axe réel, alors on peut obtenir explicitement le terme exponentiellement petit qui découle de (8).

1. Introduction

At the first Solvay Congress in 1911, Lorentz raised the question: How does a simple pendulum behave when the suspending thread is gradually shortened? This question was relevant to the quantum theory of that date.

Mathematically, this problem can be formulated as follows. Consider the equation

$$\frac{d^2u}{d\tau^2} + \omega^2(\tau)u = 0, \quad (1)$$

where

$$\begin{aligned} \omega(\tau) &\geq b_0 > 0, \quad |\omega^{(n)}(\tau)| \leq b_n \varepsilon^n, \quad \int_{-\infty}^{\infty} |\omega^{(n)}(\tau)| d\tau \leq b'_n \varepsilon^{n-1}, \\ \omega(\tau) &\rightarrow \omega(\pm\infty), \quad \omega^{(n)}(\tau) \rightarrow 0 \quad (n \geq 1) \text{ as } \tau \rightarrow \pm\infty, \end{aligned} \quad (2)$$

b_n, b'_n being constants, and $\varepsilon > 0$ being a small parameter. Let

$$H(\tau) = \frac{1}{\omega} \left[\omega^2 u^2 + \left(\frac{du}{d\tau} \right)^2 \right] = \omega u^2 + \frac{1}{\omega} \left(\frac{du}{d\tau} \right)^2. \quad (3)$$

It was conjectured that

$$H(\infty) - H(-\infty) = O(\varepsilon^n) \quad (4)$$

for every $n \geq 0$. This conjecture was first proved by Littlewood [1] in 1963, but his proof is quite complicated; see [3, p. 214, line 7].

The problem was later re-investigated by Meyer [2] and Wasow [5] in 1973. Set

$$\omega(\tau) = \phi(\varepsilon\tau), \quad (5)$$

and make the change of time scale

$$t = \varepsilon\tau.$$

Eq. (1) then becomes

$$\varepsilon^2 \ddot{u} + \phi^2(t)u = 0, \quad (6)$$

where dot “.” denotes derivative with respect to t (i.e., $\dot{u} = du/dt$). Define

$$r^2(t, \varepsilon) = \phi u^2 + \varepsilon^2 \phi^{-1}(\dot{u})^2. \quad (7)$$

Under conditions similar to (2), Wasow proved that

$$r^2(\infty, \varepsilon) - r^2(-\infty, \varepsilon) = O(\varepsilon^n) \quad (8)$$

for all $n \geq 0$, which is of course equivalent to Littlewood's result stated in (4). Wasow's approach is to first convert Eq. (6) into a system of first order equations, and then work on a Riccati equation after some matrix transformations.

In this Note, we show that after a Liouville transformation, (8) can be established by simply applying the well-known WKB approximation. We also show that if ϕ is analytic in the complex t -plane with a zero or a branch point off the real axis, then one can derive explicitly the exponentially small term that is implied by (8).

2. The WKB approximation

We begin by first recalling the following definition introduced by Wasow [5]. An infinitely differentiable real or complex function f of t is called *gentle* if

$$f^{(n)}(t) \in L^1(-\infty, \infty), \quad n = 0, 1, 2, \dots.$$

By integrating the derivatives of f , it is readily seen that

$$\lim_{t \rightarrow \pm\infty} f^{(n)}(t) = 0, \quad n = 0, 1, 2, \dots$$

Like Wasow, we make the same hypotheses on the function $\phi(t)$ in (6); namely, (i) ϕ is a positive function of t in $-\infty < t < \infty$, (ii) $\lim_{t \rightarrow \infty} \phi(t)$ and $\lim_{t \rightarrow -\infty} \phi(t)$ exist and are positive; (iii) $d\phi/dt$ is gentle. These assumptions are also the same as in Littlewood [1].

To Eq. (6), we apply the standard Liouville transformation:

$$\xi = \int^t \phi(s) ds, \quad W(\xi) = \{\phi(t)\}^{1/2} u; \quad (9)$$

see [4], pp. 190–191, Eqs. (1.03) and (1.05). The new equation reads

$$\frac{d^2 W}{d\xi^2} = \left\{ -\frac{1}{\varepsilon^2} + \Psi(\xi) \right\} W, \quad (10)$$

where

$$\Psi(\xi) = -\frac{1}{\phi^{3/2}} \frac{d^2}{dt^2} \left\{ \frac{1}{\phi^{1/2}} \right\}. \quad (11)$$

Using the hypotheses, we readily verify that $\Psi(\xi)$ is a gentle function of ξ . From (7) and (9), we also have

$$r^2(t, \varepsilon) = W^2(\xi) + \varepsilon^2 \left(\frac{dW}{d\xi} \right)^2 + \frac{\varepsilon^2}{4\phi^4} \left(\frac{d\phi}{dt} \right)^2 W^2 - \frac{\varepsilon^2}{\phi^2} \frac{d\phi}{dt} \frac{dW}{d\xi}. \quad (12)$$

Set

$$r_W^2(\xi, \varepsilon) = W^2(\xi) + \varepsilon^2 \left(\frac{dW}{d\xi} \right)^2. \quad (13)$$

Since $d\phi/dt$ is gentle, and since W and $dW/d\xi$ are bounded, we have from (12)

$$r^2(\infty, \varepsilon) - r^2(-\infty, \varepsilon) = r_W^2(\infty, \varepsilon) - r_W^2(-\infty, \varepsilon). \quad (14)$$

We now work with Eq. (10), and note that this is exactly the equation studied in Section 3, Chapter 10, of the definitive book by Olver [4]. With u there being replaced by i/ε , one readily obtains the two linearly independent solutions

$$W(\xi, \varepsilon) = e^{i\xi/\varepsilon} p(\xi, \varepsilon), \quad \bar{W}(\xi, \varepsilon) = e^{-i\xi/\varepsilon} \bar{p}(\xi, \varepsilon), \quad (15)$$

where bar “ $-$ ” denotes the complex conjugate and $p(\xi, \varepsilon)$ has the asymptotic expansion

$$p(\xi, \varepsilon) \sim \sum_{n=0}^{\infty} A_n(\xi) \varepsilon^n \quad \text{as } \varepsilon \rightarrow 0^+. \quad (16)$$

The coefficient functions $A_n(\xi)$ satisfy the recursive formula

$$A'_{n+1}(\xi) = -\frac{1}{2i} [A''_n(\xi) - A_n(\xi) \Psi(\xi)], \quad (17)$$

with $A_0(\xi) = 1$. This recurrence relation also implies that for each $n \geq 0$, $A'_n(\xi)$ is gentle. Furthermore, since $A'_n(\pm\infty) = 0$, we have

$$p'(\pm\infty, \varepsilon) = O(\varepsilon^n), \quad \bar{p}'(\pm\infty, \varepsilon) = O(\varepsilon^n) \quad (18)$$

for all $n \geq 0$; i.e., $p'(\pm\infty, \varepsilon)$ and $\bar{p}'(\pm\infty, \varepsilon)$ are asymptotically equal to zero with respect to the sequence $\{\varepsilon^n\}_{n \geq 0}$. To have two real-valued solutions, we let

$$\begin{aligned} W_1(\xi, \varepsilon) &= \frac{1}{2} [e^{i\xi/\varepsilon} p(\xi, \varepsilon) + e^{-i\xi/\varepsilon} \bar{p}(\xi, \varepsilon)], \\ W_2(\xi, \varepsilon) &= \frac{1}{2i} [e^{i\xi/\varepsilon} p(\xi, \varepsilon) - e^{-i\xi/\varepsilon} \bar{p}(\xi, \varepsilon)]. \end{aligned}$$

In matrix form, we can write them as

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = R \begin{pmatrix} e^{i\xi/\varepsilon} p(\xi, \varepsilon) \\ e^{-i\xi/\varepsilon} \bar{p}(\xi, \varepsilon) \end{pmatrix},$$

where

$$R = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2i} & -\frac{1}{2i} \end{pmatrix}.$$

Any real-valued solution of (10) can be expressed as

$$W = (C_1, C_2) \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = (C_1, C_2) R \begin{pmatrix} e^{i\xi/\varepsilon} p(\xi, \varepsilon) \\ e^{-i\xi/\varepsilon} \bar{p}(\xi, \varepsilon) \end{pmatrix}.$$

Clearly,

$$W^2 = WW^T = (C_1, C_2) R \begin{pmatrix} e^{2i\xi/\varepsilon} p^2 & p\bar{p} \\ p\bar{p} & e^{-2i\xi/\varepsilon} \bar{p}^2 \end{pmatrix} R^T \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

and

$$W^2 + \varepsilon^2 W'(W')^T = (C_1, C_2) R \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} R^T \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad (19)$$

where

$$\begin{aligned} w_{11} &= 2\varepsilon ie^{2i\xi/\varepsilon} p' + \varepsilon^2 e^{2i\xi/\varepsilon} (p')^2, & w_{22} &= \bar{w}_{11}, \\ w_{12} &= w_{21} = 2p\bar{p} + \varepsilon ip\bar{p}' - \varepsilon i\bar{p}p' + \varepsilon^2 p'\bar{p}'. \end{aligned} \quad (20)$$

To simplify w_{12} , we need the following result.

Lemma. We have

$$p\bar{p} = C + \frac{\varepsilon i}{2}(p'\bar{p} - p\bar{p}'),$$

where C is a constant.

Proof. Since W in (15) is a solution of (10), the functions p and \bar{p} satisfy the equations

$$\begin{aligned} \varepsilon p'' + 2ip' &= \varepsilon\Psi(\xi)p, \\ \varepsilon\bar{p}'' - 2i\bar{p}' &= \varepsilon\Psi(\xi)\bar{p}. \end{aligned} \quad (21)$$

Multiplying the first equation by \bar{p} and the second by p , we have upon subtraction

$$\varepsilon(\bar{p}p'' - p\bar{p}'') + 2i(p'\bar{p} + \bar{p}'p) = 0.$$

An integration gives

$$\varepsilon(p'\bar{p} - p\bar{p}') + 2i p\bar{p} = \text{Constant},$$

which is exactly the statement of the lemma. \square

By the lemma, the entries w_{12} and w_{21} in (19) can be written as

$$w_{12} = w_{21} = 2C + \varepsilon^2 p'\bar{p}' \quad (22)$$

On account of (18), (20) and (22), we obtain

$$w_{i,j}(\infty) - w_{i,j}(-\infty) = O(\varepsilon^n), \quad i, j = 1, 2,$$

for all integers $n \geq 0$. Wasow's result in (8) now follows from (13), (14) and (19).

3. Exponential asymptotics

Wasow's asymptotic expression clearly suggests that the difference $r^2(\infty, \varepsilon) - r^2(-\infty, \varepsilon)$ is exponentially small. However, the size of the exponential term is not known. In this section, we show that if $\phi(t)$ is an analytic function in the complex t -plane, then an explicit exponentially-small estimate for this difference can be given.

From (21), we have

$$e^{2i\xi/\varepsilon} p'(\xi) = \int_{-\infty}^{\xi} e^{2is/\varepsilon} p(s) \Psi(s) ds.$$

Put

$$P(\xi, \varepsilon) = e^{2i\xi/\varepsilon} p'(\xi). \quad (23)$$

Then

$$P(\infty, \varepsilon) = \int_{-\infty}^{\infty} e^{2is/\varepsilon} p(s) \Psi(s) ds. \quad (24)$$

We shall show that $P(\infty, \varepsilon)$ and its complex conjugate $\bar{P}(\infty, \varepsilon)$ are both exponentially small.

Now, assume that $\phi(t)$ has a zero in the upper half plane, say at t_0 . Near $t = t_0$, we assume that

$$\phi(t) = (t - t_0)^\beta [a_0 + a_1(t - t_0) + a_2(t - t_0)^2 + \dots],$$

where β is any complex number with $\operatorname{Re} \beta > 0$. Note that t_0 could be a branch point.

Since $\phi(t)$ is a real-valued function for $t \in (-\infty, \infty)$, we have

$$\phi(t_0) = \phi(\bar{t}_0) = 0.$$

Since $\phi(t)$ is real when t is real, without loss of generality, we may suppose that

$$\operatorname{Im} \int_0^{t_0} \phi(v) dv > 0.$$

Otherwise, we can replace t_0 by \bar{t}_0 . Set

$$s_0 = \int_0^{t_0} \phi(v) dv.$$

From (24), we have

$$P(\infty, \varepsilon) = e^{2is_0/\varepsilon} \int_{-\infty}^{\infty} e^{2i(s-s_0)/\varepsilon} p(s) \Psi(s) ds. \quad (25)$$

Since $\operatorname{Im} s_0 > 0$, the factor $e^{2is_0/\varepsilon}$ is exponentially small. We further assume that the integration path can be deformed into a loop, which starts from $s_0 + i\infty$, encircles $s = s_0$ once in a positive sense, and returns to the starting point. If t_0 is a branch point, then the vertical line joining s_0 and $s_0 + i\infty$ is a branch cut, and the loop embraces the cut. Such a contour is known as the Hankel loop. An asymptotic expansion can now be derived for $P(\infty, \varepsilon)$ by using Barnes' lemma (Watson's lemma for loop integrals); see [6, p. 48] or [4, p. 118].

In view of (23), we can rewrite the entries w_{ij} given in (20) as

$$\begin{aligned} w_{11} &= \bar{w}_{22} = 2\varepsilon i P(\xi, \varepsilon) + \varepsilon^2 p' P(\xi, \varepsilon), \\ w_{12} &= w_{21} = 2C + \varepsilon^2 e^{-2i\xi/\varepsilon} \bar{p}' e^{2i\xi/\varepsilon} p' = 2C + \varepsilon^2 |P(\xi, \varepsilon)|^2. \end{aligned}$$

From (13) and (19), it follows that

$$r_W^2(\infty, \varepsilon) - r_W^2(-\infty, \varepsilon) = O(\varepsilon P(\infty, \varepsilon)).$$

The exponentially small term is included in $P(\infty, \varepsilon)$; see (25).

If ϕ has many zeros or branch points in the upper half plane, say t_0, t_1, t_2, \dots , then we can always order them so that

$$\operatorname{Im} t_0 \leq \operatorname{Im} t_1 \leq \operatorname{Im} t_2 \leq \dots,$$

and make an asymptotic evaluation involving each loop around one of these points separately; see the idea in [6, pp. 354–355].

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