

Probability Theory
The Bessel ratio distribution

Saralees Nadarajah^a, Samuel Kotz^b

^a School of Mathematics, University of Manchester, Manchester M60 1QD, UK

^b Department of Engineering Management and Systems Engineering, George Washington University, Washington, DC 20052, USA

Received 3 January 2006; accepted after revision 31 July 2006

Presented by Marc Yor

Abstract

Let X and Y be two random variables; then the exact distribution of the ratio X/Y is derived when X and Y are independent Bessel function random variables. *To cite this article: S. Nadarajah, S. Kotz, C. R. Acad. Sci. Paris, Ser. I 343 (2006).*
© 2006 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Résumé

Distribution du rapport de deux probabilités de type Bessel. Soient X et Y deux variables aléatoires ; on en déduit la valeur du rapport X/Y dans le cas où X et Y sont des variables aléatoires dont les densités de probabilités sont de type Bessel. *Pour citer cet article : S. Nadarajah, S. Kotz, C. R. Acad. Sci. Paris, Ser. I 343 (2006).*
© 2006 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

For given random variables X and Y , the distribution of the ratio X/Y is of interest in biological and physical sciences, econometrics, and ranking and selection. Examples include Mendelian inheritance ratios in genetics, mass to energy ratios in nuclear physics, target to control precipitation in meteorology, and inventory ratios in economics. The distribution of X/Y has been studied by several authors. This Note represents the first study of the exact distribution of $Z = X/Y$ when X and Y are independent Bessel function random variables with pdfs

$$f_X(x) = \frac{|x|^m}{\sqrt{\pi} 2^m \Gamma(m + 1/2)} K_m(|x|), \quad (1)$$

and

$$f_Y(y) = \frac{|y|^n}{\sqrt{\pi} 2^n \Gamma(n + 1/2)} K_n(|y|), \quad (2)$$

E-mail addresses: saralees.nadarajah@manchester.ac.uk (S. Nadarajah), kotz@seas.gwu.edu (S. Kotz).

respectively, for $-\infty < x < \infty$, $-\infty < y < \infty$, $m > 1$ and $n > 1$, where

$$K_\nu(x) = \frac{\sqrt{\pi} x^\nu}{2^\nu \Gamma(\nu + 1/2)} \int_1^\infty (t^2 - 1)^{\nu-1/2} \exp(-xt) dt$$

is the modified Bessel function of the third kind. It is well-known that Bessel function random variables of the above type can be characterized as linear combinations of independent chi-squared random variables (Bhattacharyya, [1]). Thus, a ratio of the form $Z = X/Y$ amounts to taking a ratio of linear combinations of chi-squared random variables. Ratios of this type are part of von Neumann's [5] test statistics (mean square successive difference divided by the variance). These ratios appear in various two-stage tests (Toyoda and Ohtani, [8]). They are also used in tests on structural coefficients of a multivariate linear functional relationship model (details in Chaubey and Nur Enayet Talukder [2] and Provost and Rudiuk [6]).

Another motivation for considering $Z = X/Y$ comes from practical applications that the Bessel function distribution has found in a variety of areas that range from image and speech recognition and ocean engineering to finance. It is rapidly becoming a distribution of first choice whenever 'something' with heavier than Gaussian tails is observed in the data. In many of the application areas, one would be interested in ratios of Bessel function random variables. Some examples are:

1. in communication theory, X and Y could represent the random noise corresponding to two different signals;
2. in ocean engineering, X and Y could represent distributions of navigation errors;
3. in finance, X and Y could represent distributions of log-returns of two different commodities;
4. in image and speech recognition, X and Y could represent 'input' distributions.

For further discussion of applications, the reader is referred to Kotz et al. [4].

Since the distribution of $Z = X/Y$ is symmetric around zero it will be sufficient to derive its form for $z > 0$. The calculations will involve the Gauss hypergeometric function defined by

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}, \quad (3)$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. The series in (3) converges in the unit circle $|x| < 1$ with a branch point at $x = 1$. The properties of (3) can be found in Prudnikov et al. [7] and Gradshteyn and Ryzhik [3].

2. The ratio distribution

Theorem 1 derives an explicit expression for the pdf of X/Y in terms of the Gauss hypergeometric function:

Theorem 1. *Suppose X and Y are distributed according to (1) and (2), respectively. The pdf of $Z = X/Y$ can be expressed as*

$$f_Z(z) = \frac{z^{-2n-2} \Gamma(m+1) \Gamma(n+1)}{\pi(m+n+1) \Gamma(m+1/2) \Gamma(n+1/2)} {}_2F_1(m+n+1, n+1; m+n+2; 1-z^{-2}), \quad (4)$$

for $z > 0$.

Proof. The pdf of $Z = X/Y$ for $z > 0$ can be expressed as

$$\begin{aligned} f_Z(z) &= 2 \int_0^\infty y f_X(yz) f_Y(y) dy = 2 \int_0^\infty y \frac{(yz)^m}{\sqrt{\pi} 2^m \Gamma(m+1/2)} K_m(yz) \frac{y^n}{\sqrt{\pi} 2^n \Gamma(n+1/2)} K_n(y) dy \\ &= \frac{z^m I(m, n, z)}{\pi 2^{m+n-1} \Gamma(m+1/2) \Gamma(n+1/2)}, \end{aligned} \quad (5)$$

where $I(m, n, z)$ denotes the integral $I(m, n, z) = \int_0^\infty y^{m+n+1} K_m(yz) K_n(y) dy$. The result of the theorem follows by direct application of Eq. (6.576.4) in Gradshteyn and Ryzhik [3] to calculate $I(m, n, z)$. \square

Using special properties of the Gauss hypergeometric function, one can derive other equivalent forms and elementary forms for the pdf of $Z = X/Y$. This is illustrated in the corollaries below.

Corollary 1. *The pdf given by (4) can be expressed in the equivalent forms:*

$$f_Z(z) = \frac{z^{2m}\Gamma(m+1)\Gamma(n+1)}{\pi(m+n+1)\Gamma(m+1/2)\Gamma(n+1/2)} {}_2F_1(m+n+1, m+1; m+n+2; 1-z^2), \tag{6}$$

$$f_Z(z) = \frac{\Gamma(m+1)\Gamma(n+1)}{\pi(m+n+1)\Gamma(m+1/2)\Gamma(n+1/2)} {}_2F_1(n+1, 1; m+n+2; 1-z^2), \tag{7}$$

and

$$f_Z(z) = \frac{z^{-2}\Gamma(m+1)\Gamma(n+1)}{\pi(m+n+1)\Gamma(m+1/2)\Gamma(n+1/2)} {}_2F_1(1, m+1; m+n+2; 1-z^{-2}). \tag{8}$$

Note that (6) and (7) hold for $-\sqrt{2} < z < \sqrt{2}$ while (8) holds for $z < -1/\sqrt{2}$ or $z > 1/\sqrt{2}$.

Proof. Apply the three transformation formulas of Eq. (9.131.1) in Gradshteyn and Ryzhik [3] to the hypergeometric term in (4). \square

Corollary 2. *If $m \geq 2$ and $n \geq 2$ are integers then (4) can be reduced to the elementary form:*

$$f_Z(z) = \frac{\Gamma(m+1)\Gamma(m+n+1)}{\pi(z^2-1)\Gamma(m+1/2)\Gamma(n+1/2)} \times \left[\sum_{k=1}^n \frac{(n-k)!(1-z^2)^{1-k}}{(m+n+1-k)!} + \frac{(1-z^2)^{1-n}}{m!z^2} \left\{ -2(1-z^{-2})^{-m-1} \log z + \sum_{k=1}^m \frac{(1-z^{-2})^{-k}}{m+1-k} \right\} \right].$$

Proof. Apply Eq. (7.3.1.128) in Prudnikov et al. [7, volume 3] to the hypergeometric term in (8). \square

Corollary 3. *If $m - 1/2 \geq 1$ and $n - 1/2 \geq 1$ are integers then (4) can be reduced to the elementary form:*

$$f_Z(z) = \frac{z^{-2}(1-z^2)^{-m-n-1}\Gamma(m+1)\Gamma(n+1)}{\pi(m+n+1)(-m-1)_{m+n+2}\Gamma(m+1/2)\Gamma(n+1/2)} \times \left[\Gamma(-m)z^{2m+2} + \sum_{k=1}^{m+n+1} (-m-1)_k (-1)^k (1-z^{-2})^{k-1} z^{2k} \right].$$

Proof. Apply Eq. (7.3.1.124) in Prudnikov et al. [7, volume 3] to the hypergeometric term in (8). \square

Note that Corollaries 2 and 3 provide the particular forms of (4) when m & n take integer and half-integer values, respectively. Both type of forms are elementary. The shape of $f_Z(z)$ can be determined as follows by using the representations in Corollary 1. Using the fact,

$${}_2F_1(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)},$$

it follows from (7) that

$$f_Z(z) \rightarrow \frac{\Gamma(m)\Gamma(n+1)}{\pi\Gamma(m+1/2)\Gamma(n+1/2)}$$

as $z \rightarrow 0$. On the other hand, it follows from (8) that

$$f_Z(z) \sim \frac{\Gamma(m+1)\Gamma(n)z^{-2}}{\pi\Gamma(m+1/2)\Gamma(n+1/2)},$$

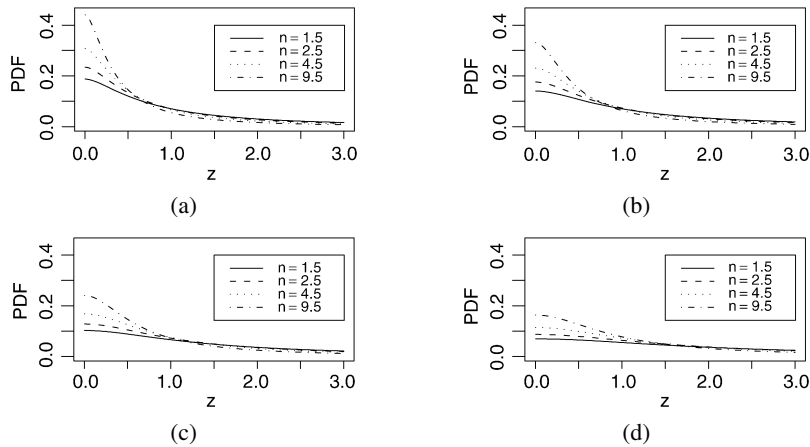


Fig. 1. Plots of the pdf of (4) for $b = 1$, $\beta = 1$ and (a): $m = 1.5$; (b): $m = 2.5$; (c): $m = 4.5$; and (d): $m = 9.5$.

as $z \rightarrow \infty$. Using the fact,

$$\frac{\partial}{\partial z} {}_2F_1(\alpha, \beta; \gamma; z) = \frac{\alpha\beta}{\gamma} {}_2F_1(\alpha + 1, \beta + 1; \gamma + 1; z), \quad (9)$$

and (7), the derivative of $f_Z(z)$ with respect to z for $z < \sqrt{2}$ can be calculated as

$$f'_Z(z) = -\frac{2z\Gamma(m+1)\Gamma(n+2){}_2F_1(n+2, 2; m+n+3; 1-z^2)}{\pi(m+n+1)(m+n+2)\Gamma(m+1/2)\Gamma(n+1/2)} < 0. \quad (10)$$

For $z > \sqrt{2}$, using (8) and (9), one has

$$f'_Z(z) = \frac{2z^{-3}\Gamma(m+1)\Gamma(n+1)}{\pi(m+n+1)\Gamma(m+1/2)\Gamma(n+1/2)} \times \left\{ \frac{m+1}{(m+n+2)z^2} {}_2F_1(2, m+2; m+n+3; 1-z^{-2}) - {}_2F_1(1, m+1; m+n+2; 1-z^{-2}) \right\}. \quad (11)$$

Now, using the recurrence relation that

$$\gamma {}_2F_1(\alpha, \beta; \gamma; z) + (\beta - \gamma) {}_2F_1(\alpha + 1, \beta; \gamma + 1; z) = \beta(1 - z) {}_2F_1(\alpha + 1, \beta + 1; \gamma + 1; z),$$

Eq. (11) can be reduced to the simple form:

$$f'_Z(z) = -\frac{2z^{-3}\Gamma(m+1)\Gamma(n+2){}_2F_1(2, m+1; m+n+3; 1-z^{-2})}{\pi(m+n+1)(m+n+2)\Gamma(m+1/2)\Gamma(n+1/2)} < 0. \quad (12)$$

Thus, (10) and (12) show that $f_Z(z)$ is a strictly decreasing function for all $z > 0$. This is confirmed by Fig. 1, where we have sketched the pdf of X/Y for a range of values of m and n .

References

- [1] B.C. Bhattacharyya, The use of McKay's Bessel function curves for graduating frequency distributions, *Sankhyā* 6 (1942) 175–182.
- [2] Y.P. Chaubey, A.B.M. Nur Enayet Talukder, Exact moments of a ratio of two positive quadratic forms in normal variables, *Communications in Statistics—Theory and Methods* 12 (1983) 675–679.
- [3] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products*, sixth ed., Academic Press, San Diego, 2000.
- [4] S. Kotz, T.J. Kozubowski, K. Podgorski, *The Laplace Distribution and Generalizations: A Revisit with Applications to Communications, Economics, Engineering and Finance*, Birkhäuser, Boston, 2001.
- [5] J. von Neumann, Distribution of the ratio of the mean square successive difference to the variance, *Annals of Mathematical Statistics* 12 (1941) 367–395.
- [6] S.B. Provost, E.M. Rudiuk, The exact density function of the ratio of two dependent linear combinations of chi-square variable, *Annals of the Institute of Statistical Mathematics* 46 (1994) 557–571.
- [7] A.P. Prudnikov, Y.A. Brychkov, O.I. Marichev, *Integrals and Series*, vols. 1–3, Gordon and Breach Science Publishers, Amsterdam, 1986.
- [8] T. Toyoda, K. Ohtani, Testing equality between sets of coefficients after a preliminary test for equality of disturbance variances in two linear regressions, *Journal of Econometrics* 31 (1986) 67–80.