## Differential Geometry

Einstein solvmanifolds and graphs

Hamid-Reza Fanaï ${ }^{\text {ab, }, 1}$<br>${ }^{\text {a }}$ Department of Mathematical Sciences, Sharif University of Technology, P.O. Box 11365-9415, Tehran, Iran<br>${ }^{\mathrm{b}}$ Institute for Studies in Theoretical Physics and Mathematics (IPM), Tehran, Iran

Received 16 October 2006; accepted 8 November 2006
Available online 8 December 2006
Presented by Étienne Ghys


#### Abstract

In this Note, we obtain Einstein solvmanifolds using Abelian extension of two-step nilpotent Lie algebras associated with graphs. To cite this article: H.-R. Fanaï, C. R. Acad. Sci. Paris, Ser. I 344 (2007). © 2006 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.


## Résumé

Solvariétés d'Einstein et graphes. Dans cette Note, nous obtenons des solvariétés d'Einstein par extension abélienne des algèbres de Lie nilpotentes de rang deux associées à des graphes. Pour citer cet article : H.-R. Fanaï, C. R. Acad. Sci. Paris, Ser. I 344 (2007).
© 2006 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

## 1. Introduction

Our aim in this Note is the construction of examples of homogeneous Einstein manifolds of negative scalar curvature. The classical examples of Einstein metrics of negative scalar curvature are the symmetric spaces of non-compact type. A number of other examples are known, e.g. [1,3-5,7,8]. A Riemannian solvmanifold is a simply connected solvable Lie group $S$ together with a left invariant Riemannian metric $g$. All known examples of homogeneous Einstein manifolds of negative scalar curvature are isometric to Riemannian solvmanifolds. In these examples, such a solvable Lie group is a semi-direct product of an Abelian group $A$ with a nilpotent normal subgroup $N$, the nilradical. We consider the class of solvmanifolds for which $N$ is two-step nilpotent associated with a graph as introduced in [2], and $A$ is one-dimensional. The left invariant Riemannian metric $g$ on $S$ defines an inner product $\langle$,$\rangle on the Lie algebra$ $\mathfrak{s}$ of $S$. We will refer to a Lie algebra endowed with an inner product as a metric Lie algebra.

Recently in [2], two-step nilpotent Lie algebras attached to graphs are considered, where their group of Lie automorphisms have been determined. In this Note, we are interested in such a (metric) two-step nilpotent Lie algebra $\mathfrak{n}$ and study whether there is a metric solvable extension of $(\mathfrak{n},\langle\rangle$,$) which is Einstein. More precisely, we construct a$ metric solvable Lie algebra $(\mathfrak{s}=\mathfrak{a} \oplus \mathfrak{n},\langle\rangle$,$) where \mathfrak{a}$ is one-dimensional orthogonal to $\mathfrak{n}$ and the Lie bracket and the

[^0]inner product on $\mathfrak{s}$ restricted to $\mathfrak{n}$ are those of $\mathfrak{n}$ and describe necessary and sufficient conditions, in terms of the graph, for the manifold $\mathfrak{s}$ to be Einstein. This is contained in Theorem 1 in the last section. We recall the construction with graphs introduced in [2] and some notations in the next section.

## 2. Preliminaries

Let $(V, E)$ be finite graph, where $V$ is the set of vertices and $E$ is the set of edges; equivalently $E$ is a collection of unordered pairs of distinct vertices; the unordered pairs will be written in the form $\alpha \beta$, where $\alpha, \beta \in V$. Let $\mathfrak{v}$ be a vector space with $V$ as a basis. Let $\mathfrak{z}$ be the subspace of $\wedge^{2} \mathfrak{v}$ the second exterior power of $\mathfrak{v}$, spanned by $\{\alpha \wedge \beta: \alpha, \beta \in V, \alpha \beta \in E\}$. Let $\mathfrak{n}=\mathfrak{z} \oplus \mathfrak{v}$. Stipulating the conditions that for any $\alpha, \beta \in V,[\alpha, \beta]=\alpha \wedge \beta$ if $\alpha \beta \in E$ and 0 otherwise, and $[Z, X]=0$ for all $Z \in \mathfrak{z}$ and $X \in \mathfrak{v}$ determines a unique Lie algebra structure on $\mathfrak{n}$ (see [2]). Clearly $\mathfrak{n}$ is a two-step nilpotent Lie algebra. Since without loss of generality, we can assume that the graph is a connected graph hence the center of $\mathfrak{n}$ is $\mathfrak{z}=[\mathfrak{n}, \mathfrak{n}]$. We endow $\mathfrak{n}$ with an inner product $\langle$,$\rangle such that \mathfrak{z} \perp \mathfrak{v}$ and two orthonormal bases of $\mathfrak{v}$ and $\mathfrak{z}$ are $V=\left\{X_{1}, \ldots, X_{k}\right\}$ and $\{\alpha \wedge \beta: \alpha, \beta \in V, \alpha \beta \in E\}$ respectively. Write the latter basis of $\mathfrak{z}$ as $\left\{Z_{1}, \ldots, Z_{r}\right\}$ where each element of the basis is an edge $X_{i} X_{j}$ with $i<j$. We can then define a linear map $J: \mathfrak{z} \rightarrow \mathfrak{s o}(\mathfrak{v})$ by

$$
\langle J(Z) X, Y\rangle=\langle[X, Y], Z\rangle
$$

for $X, Y \in \mathfrak{v}$ and $Z \in \mathfrak{z}$. The endomorphism $J^{*} J$ of $\mathfrak{z}$ is represented by the $r \times r$ matrix $\left(-\operatorname{tr}\left({ }^{t} J\left(Z_{i}\right) J\left(Z_{j}\right)\right)\right)_{i, j}$.
Consider now a metric solvable extension ( $\mathfrak{s}=\mathbb{R} A \oplus \mathfrak{n},\langle$,$\rangle ) of \mathfrak{n}$ where the norm of $A$ is one. We let $f:=$ $\operatorname{ad}_{A}: \mathfrak{n} \rightarrow \mathfrak{n}$. We can suppose that $\operatorname{tr} f>0$. According to [5] and [6], if the solvmanifold is Einstein, then the matrix representation of $f$ in the above bases is in the following form:

$$
f=\left(\begin{array}{cc}
B_{r} & 0 \\
0 & D_{k}
\end{array}\right) .
$$

In fact the solvmanifold $\mathfrak{s}$ is Einstein if and only if there exists a (negative) constant $\mu$ such that we have the following relations (see [5,6]):
( $\left.\mathrm{E}_{1}\right)-(\operatorname{tr} f) B+\frac{1}{4} J^{*} J=\mu \mathrm{Id}_{r}$,
$\left(\mathrm{E}_{2}\right)-(\operatorname{tr} f) D+\frac{1}{2} \sum_{i=1}^{r} J^{2}\left(Z_{i}\right)=\mu \mathrm{Id}_{k}$,
$\left(\mathrm{E}_{3}\right) J(B(\cdot))=J(\cdot) D+D J(\cdot)$.
In the next section we consider these relations.

## 3. Result

Suppose that the relations $\left(\mathrm{E}_{1}\right),\left(\mathrm{E}_{2}\right)$ and $\left(\mathrm{E}_{3}\right)$ are satisfied for $f$ (for a certain negative constant $\left.\mu\right)$ and thus the solvable extension is Einstein. We are looking for conditions imposed on the graph by these relations. The matrix representation of $J\left(Z_{i}\right), 1 \leqslant i \leqslant r$, is easy to obtain. If the edge $Z_{i}$ as a unordered pair is equal to $X_{j} X_{j^{\prime}}$ with $1 \leqslant j<j^{\prime} \leqslant k$, then the only nonzero entries of $J\left(Z_{i}\right)$ are $J\left(Z_{i}\right)_{j j^{\prime}}=-1$ and $J\left(Z_{i}\right)_{j^{\prime} j}=1$. Hence $J^{*} J=2 \operatorname{Id}_{r}$ and from ( $\mathrm{E}_{1}$ ) we have $B=\lambda \operatorname{Id}_{r}$ for a constant $\lambda$. On the other hand, $\sum_{i=1}^{r} J^{2}\left(Z_{i}\right)=-\operatorname{diag}\left(d_{1}, \ldots, d_{k}\right)$ where $d_{j}$ is the degree of the vertex $X_{j}, 1 \leqslant j \leqslant k$. This implies from $\left(\mathrm{E}_{2}\right)$ that $D=\operatorname{diag}\left(d_{1}^{\prime}, \ldots, d_{k}^{\prime}\right)$ for certain constant $d_{j}^{\prime}$, $1 \leqslant j \leqslant k$, which are determined by the degrees of the vertices in the graph. As $B$ is a multiple of identity and $D$ is diagonal, it follows from ( $\mathrm{E}_{3}$ ) that $d_{i}^{\prime}+d_{j}^{\prime}=\lambda$ whenever $X_{i} X_{j} \in E$. So the neighbors of a vertex have the same degree. Hence if there is a walk in the graph from a vertex $x$ to a vertex $y$ which contains an even number of edges, i.e. an even walk, then the vertices $x$ and $y$ have the same degree. Now define an equivalence relation on the vertices in the following way: the vertex $X_{i}$ is related to $X_{j}$ iff there is an even walk in the graph from $X_{i}$ to $X_{j}$. Consider the equivalence classes. We have the following different cases:

Case 1. There is only one equivalence class. It means that the graph is a regular graph, i.e. all the vertices have the same degree, say $d$. This is actually possible and we can construct a solvable extension of $\mathfrak{n}$ which is Einstein. In fact the number of edges is equal to $r=\frac{k d}{2}$ and it is easy to see that the relations $\left(\mathrm{E}_{1}\right),\left(\mathrm{E}_{2}\right)$ and $\left(\mathrm{E}_{3}\right)$ are satisfied with $B=\frac{1}{\operatorname{tr} f}(1+d) \operatorname{Id}_{r}, D=\frac{1}{2 \operatorname{tr} f}(1+d) \operatorname{Id}_{k}$ where $\operatorname{tr} f=\sqrt{(1+d)\left(r+\frac{k}{2}\right)}$ and $\mu=-\frac{1}{2}-d$.

Case 2 . There are more than two equivalence classes. This is actually impossible when the graph is connected, because in this case, three different equivalence classes $C_{1}, C_{2}$ and $C_{3}$ exist such that we have the vertices $\alpha \in C_{1}$, $\beta \in C_{2}$ with $\alpha \beta \in E$ and the vertices $\alpha^{\prime} \in C_{2}, \beta^{\prime} \in C_{3}$ with $\alpha^{\prime} \beta^{\prime} \in E$. If $\beta=\alpha^{\prime}$ then the vertices $\alpha$ and $\beta^{\prime}$ must be in the same equivalence class, which is impossible. So $\beta \neq \alpha^{\prime}$, as these vertices are in the same equivalence class $C_{2}$, there is an even walk in the graph from $\beta$ to $\alpha^{\prime}$. This leads to an even walk from $\alpha$ to $\beta^{\prime}$, hence $\alpha$ and $\beta^{\prime}$ must be in the same equivalence class, which is impossible. So the case 2 cannot occur.

Case 3. There are exactly two equivalence classes $C$ and $C^{\prime}$. As the graph is connected, there is at least one edge between $C$ and $C^{\prime}$. Now apply the argument used in the case 2 , to show that $C$ must be an independent set in the graph, i.e. no two vertices in $C$ are adjacent. Similarly $C^{\prime}$ is an independent set in the graph. This means that the graph is a bipartite graph. Recall that a graph is bipartite if the set of vertices is the union of two disjoint sets such that each edge consists of one vertex from each set. As all the vertices in an equivalence class have the same degree, we obtain that the graph is a bipartite graph with partite sets $C$ and $C^{\prime}$ such that all the vertices in $C$ have the same degree $d$ and all the vertices in $C^{\prime}$ have the same degree $d^{\prime}$. This is actually possible and we can construct a solvable extension of $\mathfrak{n}$ which is Einstein. In fact it is easy to see that the relations $\left(\mathrm{E}_{1}\right)$, $\left(\mathrm{E}_{2}\right)$ and $\left(\mathrm{E}_{3}\right)$ are satisfied with the following data where we suppose that the vertices of the graph $\left\{X_{1}, \ldots, X_{k}\right\}$ are indexed such that the $k_{1}$ first vertices are in $C\left(k_{1}=|C|\right)$ and the $k_{2}$ last vertices are in $C^{\prime}\left(k_{2}=\left|C^{\prime}\right|, k=k_{1}+k_{2}\right)$ :

$$
\begin{aligned}
& B=\frac{1}{\operatorname{tr} f\left(1+\frac{d+d^{\prime}}{2}\right) \operatorname{Id}_{r},} \\
& D=\operatorname{diag}\left(d_{1}^{\prime}, \ldots, d_{1}^{\prime}, d_{2}^{\prime}, \ldots, d_{2}^{\prime}\right), \\
& \operatorname{tr} f=\sqrt{\frac{k}{2}(1+d)\left(1+d^{\prime}\right)}, \\
& \mu=\frac{-d-d^{\prime}-1}{2}
\end{aligned}
$$

where in $D$, the diagonal entries $d_{1}^{\prime}$ (respectively $d_{2}^{\prime}$ ) are repeated $k_{1}$ times (respectively $k_{2}$ times) and

$$
d_{1}^{\prime}=\frac{1+d^{\prime}}{2 \operatorname{tr} f}, \quad d_{2}^{\prime}=\frac{1+d}{2 \operatorname{tr} f} .
$$

Note that $r=k_{1} d=k_{2} d^{\prime}$. Hence we have shown the following result:
Theorem 1. Let $(\mathfrak{n},\langle\rangle$,$) be a metric two-step nilpotent Lie algebra associated with a connected graph. Then there is$ a metric solvable extension of $\mathfrak{n}$ which is Einstein if and only if the graph is regular or a bipartite graph such that all the vertices in each partite set have the same degree.

## References

[1] D.V. Alekseevskii, Classification of quaternionic spaces with a transitive solvable group of motions, Math. USSR-Izv. 9 (1975) $297-339$.
[2] S.G. Dani, M.G. Mainkar, Anosov automorphisms on compact nilmanifolds associated with graphs, Trans. Amer. Math. Soc. 357 (2005) 2235-2251.
[3] H.-R. Fanaï, Espaces homogènes d'Einstein non-compacts, Geom. Dedicata 80 (2000) 187-200.
[4] C.S. Gordon, M.M. Kerr, New homogeneous Einstein metrics of negative Ricci curvature, Ann. Global Anal. Geom. 19 (2001) $75-101$.
[5] J. Heber, Noncompact homogeneous Einstein spaces, Invent. Math. 133 (1998) 279-352.
[6] D. Kass-Hengesch, Exemples de variétés homogènes d’Einstein à courbure scalaire négative, Thèse de Doctorat, Université Henri PoincaréNancy I, France, 1996.
[7] M. Lanzendorf, Einstein metrics with nonpositive sectional curvature on extensions of Lie algebra of Heisenberg type, Geom. Dedicata 66 (1997) 187-202.
[8] T. Wolter, Einstein metrics on solvable groups, Math. Z. 206 (1991) 457-471.


[^0]:    E-mail address: fanai@ sharif.ac.ir (H.-R. Fanaï).
    1 This research was in part supported by a grant from IPM (No. 85370033).

    1631-073X/\$ - see front matter © 2006 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved. doi:10.1016/j.crma.2006.11.010

