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Optimal Control

Description of all privileged coordinates in the homogeneous approximation problem for nonlinear control systems

Grigory Sklyar [a], Svetlana Ignatovich [b]

[a] *University of Szczecin, Wielkopolska str., 15, 70-451, Szczecin, Poland*
[b] *Kharkov National University, Svoboda sqr. 4, Kharkov, 61077, Ukraine*

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Abstract

In a homogeneous approximation problem for affine control systems, privileged coordinates are those in which the system takes a 'triangular' form allowing one to find an approximating system. We give the necessary and sufficient conditions for coordinates to be privileged. We apply an algebraic technique based on the series representation of affine control systems. ***To cite this article: G. Sklyar, S. Ignatovich, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Description de toutes les coordonnées privilégiées dans le problème d'approximation homogène pour les systèmes contrôlés non linéaires. Dans le problème d'approximation homogène pour des systèmes contrôlés affines les coordonnées privilégiées sont celles dans lesquelles le système a une forme « triangulaire » qui permet de trouver un système d'approximation. Nous donnons les conditions nécessaires et suffisantes pour que des coordonnées soient privilégiées. Nous utilisons une technique algébrique basée sur la représentation par des séries de systèmes de commande affines. ***Pour citer cet article : G. Sklyar, S. Ignatovich, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***
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Nous considérons un système commandé non linéaire de la forme (1) où $X_0(x), \dots, X_h(x)$ sont des champs vectoriels analytiques dans un voisinage de l'origine. Le problème de Cauchy (1) donne lieu à l'opérateur point terminal $\mathcal{E}_X : [0, T] \times L_\infty([0, T], \mathbb{R}^h) \to \mathbb{R}^n$ qui applique le couple θ et $u = u(t) = (u_1(t), \dots, u_h(t))$ sur le point final $x_u(\theta)$ de la trajectoire $x_u(t)$ correspondant à la commande $u = u(t)$, $t \in [0, \theta]$. Cet opérateur est développé en série de Fliess (2) dont les termes sont les intégrales itérées $\eta_{i_1 \dots i_k}(\theta, u)$ de la forme (3) avec des coefficients vectoriels constants. Cette série converge absolument si θ est suffisamment petit et si $\|u\|_\infty \leqslant 1$. Ce développement justifie l'introduction et l'étude de l'algèbre associative libre $\mathcal{F}$ des intégrales itérées $\eta_{i_1 \dots i_k}$ sur $\mathbb{R}$, engendrée par l'ensemble $\{\eta_i \colon 0 \leqslant i \leqslant h\}$ munie de l'opération produit définie par $\eta_{i_1 \dots i_k} \vee \eta_{j_1 \dots j_s} = \eta_{i_1 \dots i_k j_1 \dots j_s}$. La contrainte donnée a priori sur la commande

E-mail addresses: sklar@sus.univ.szczecin.pl (G. Sklyar), ignatovich@ukr.net (S. Ignatovich).

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engendrèe une structure graduée dans l'algèbre, $\mathcal{F}=\sum_{k=1}^{\infty}\mathcal{F}_k$, où $\mathcal{F}_k=\mathrm{Lin}\{\eta_{i_1\ldots i_k}\colon 0\leqslant i_1,\ldots,i_k\leqslant h\}$. Les propriétés du système se ramènent aux propriétés de l'application linéaire $c:\mathcal{F}\to\mathbb{R}^n$ définie par $c(\eta_{i_1\ldots i_k})=X_{i_k}\cdots X_{i_1}(0)$. En fait, l'application c induit une structure d'idéal à droite de l'algèbre $\mathcal{F}$ [13]. Plus précisément, les éléments de l'idéal à gauche J_X engendréés par les sous-espaces $\mathcal{P}_\gamma$ de la forme (4) vérifient la propriété suivante : si $a\in J_X\cap\mathcal{F}_\gamma$, alors $c(a)\in c(\sum_{k<\gamma}\mathcal{F}_k)$. Soulignons que J_X est invariant par rapport à un changement de variables du système.

Comme conséquence, le problème de l'approximation homogène pour le système initial (points (i), (ii) de la Section 1) se réduit au problème de l'analyse des termes *principaux* de la série $\mathcal{E}_X$ dans le sens de la structure graduée donnée (items (i′), (ii′) dans la Section 2). Toute transformation $y=Q(x)$, réduisant la série $\mathcal{E}_X$ à la forme, qui permet d'extraire ces termes principaux définit ce qu'on appelle *des coordonnées privilégiées* pour le système (1). Le but de cette Note est de décrire toutes les coordonnées privilégiées.

Le théorème de Melançon et Reutenauer [9,10] sur la suite duale de la base de Poincaré–Birkhoff–Witt est à la base de nos considérations. Pour être plus précis, supposons que $\mathcal{L}$ soit une algèbre de Lie engendrée par l'ensemble $\{\eta_i\colon 0\leqslant i\leqslant h\}$ avec les crochets de Lie $[\ell_1,\ell_2]=\ell_1\vee\ell_2-\ell_2\vee\ell_1$, alors $\mathcal{F}$ est l'algèbre associative enveloppante de $\mathcal{L}$. Ensuite nous utilisons la propriété de l'application linéaire c et la structure graduée. Considérons d'abord un ensemble d'éléments de Lie homogènes (dans le sens de cette structure graduée) $\ell_1,\ldots,\ell_n$ tels que $\mathcal{L}=\mathrm{Lin}\{\ell_1,\ldots,\ell_n\}+\sum_{\gamma=1}^{\infty}\mathcal{P}_\gamma$. Supposons que la suite d'éléments homogènes $\{\ell_j\}_{j=n+1}^{\infty}$ complète cet ensemble jusqu'à une base de $\mathcal{L}$. Alors la série $\mathcal{E}_X$ peut être représentée sous la forme (6) où $\mathcal{T}$ ne contient pas de termes principaux. Par ailleurs, l'opération produit de mélange (5) mentionnée en (6) correspond à un produit usuel (ponctuel) d'intégrales itérées comme fonctionnelles de u. Ainsi, des coordonnées sont priviligiées si et seulement si elles permettent d'extraire les termes principaux de l'application $\Phi(z)$ de la forme (7).

Théorème 0.1. *L'application analytique non singulière $y=Q(x)$ définit des coordonnées privilégiées si et seulement si elle réduit l'application* (7) *sous forme « triangulaire »* (8).

En particulier, les coordonnées $y=\Phi^{-1}(x)$ sont privilégiées [5]. D'autre part, il est facile de voir, qu'au lieu de $\Phi(z)$, il est suffisant de considérer l'application $\widetilde{\Phi}(z)$ de la forme (9).

Corollaire 0.2. *L'application analytique non singulière $y=Q(x)$ définit des coordonnées privilégiées si et seulement si elle réduit l'application* (9) *sous forme « triangulaire »* (8).

Parce que $\widetilde{\Phi}(z)$ est polynomiale, alors l'application $y=Q(x)$ peut être choisie sous forme polynomiale [2,13].

1. Introduction

Let us consider a class of nonlinear affine control systems of the form:

$$\dot{x}=X_0(x)+\sum_{i=1}^{h}X_i(x)u_i,\qquad x(0)=0,\tag{1}$$

where $X_0(x),\ldots,X_h(x)$ are analytic vector fields in a neighborhood of the origin in $\mathbb{R}^n$. Below we denote system (1) by $\{X\}$ (where for simplicity we use the notation $X=(X_0,\ldots,X_h)$). We say that the control $u(t)=(u_1(t),\ldots,u_h(t))$ is *admissible* if $u(t)\in L_\infty([0,T];\mathbb{R}^h)$ and $\|u\|_\infty\leqslant 1$. The Cauchy problem (1) generates *the endpoint map* $\mathcal{E}_X:[0,T]\times L_\infty([0,T];\mathbb{R}^h)\to\mathbb{R}^n$ which maps a pair θ and $u=u(t)$ to the endpoint $x_u(\theta)$ of the trajectory $x_u(t)$ of system (1) corresponding to the control $u=u(t)$, $t\in[0,\theta]$ (we assume $T>0$ is rather small). Moreover, throughout of the Note, we consider only systems *accessible from the origin*, i.e. such systems that the set of reachable points $\{x_u(\theta)\colon \|u\|_\infty\leqslant 1, 0\leqslant\theta\leqslant T\}$ has nonempty interior in $\mathbb{R}^n$. In 1981 M. Fliess [4] proposed to represent the map $\mathcal{E}_X(\theta,u)$ as a series (which is absolutely convergent if u is admissible, $\theta\in[0,T]$, and T is rather small) of the form:

$$\mathcal{E}_X(\theta,u)=\sum_{k\geqslant 1,\ 0\leqslant i_1,\ldots,i_k\leqslant h}X_{i_k}\cdots X_{i_1}(0)\,\eta_{i_1\ldots i_k}(\theta,u),\tag{2}$$

where $\eta_{i_1\ldots i_k}(\theta,u)$ are *iterated integrals,*

$$\eta_{i_1\ldots i_k}(\theta,u)=\int_0^{\theta}\int_0^{\tau_1}\cdots\int_0^{\tau_{k-1}}u_{i_1}(\tau_1)\cdots u_{i_k}(\tau_k)\,\mathrm{d}\tau_k\cdots\mathrm{d}\tau_1,\qquad u_0(t)\equiv 1. \tag{3}$$

Close representation using *series of nonlinear power moments* appears in the time-optimal control problem [12] and in the problem of approximating of a system along a trajectory [3]. All the conclusions of the present Note can also be applied for nonlinear power moments series.

Expansion (2) gives an explicit coordinate description of the endpoint map. In terms of the map $\mathcal{E}_X(\theta,u)$, a *homogeneous approximation problem* for system (1) can be expressed as follows: given a system of the form (1), accessible from the origin, to find a *homogeneous* system $\{\widehat{X}\}$, accessible from the origin, such that the map $\mathcal{E}_{\widehat{X}}(\theta,u)$ *approximates* $\mathcal{E}_X(\theta,u)$ as $\theta\to 0$. Namely:

(i) the homogeneous system $\{\widehat{X}\}$ is accessible from the origin and, without loss of generality, satisfies the property $\mathcal{E}_{\widehat{X}}(\theta,u)=\delta_\theta\mathcal{E}_{\widehat{X}}(1,\tilde{u})$ where δ_θ is a dilation acting as $\delta_\theta z=(\theta^{\gamma_1}z_1,\ldots,\theta^{\gamma_n}z_n)$ and $\tilde{u}(t)=u(t\theta)$, $t\in[0,1]$, where $1\leqslant\gamma_1\leqslant\cdots\leqslant\gamma_n$ (this is one of the most commonly used definitions of the homogeneity);
(ii) after a certain change of variables $y=Q(x)$ in the system $\{X\}$, for any admissible $u=u(t)$ we have $\delta_\theta^{-1}(Q(\mathcal{E}_X(\theta,u))-\mathcal{E}_{\widehat{X}}(\theta,u))\to 0$ as $\theta\to 0$.

The concepts of homogeneity and approximation can be introduced and treated in a coordinate-free manner by means of the differential geometric language. Nevertheless, one returns to coordinates when constructing concrete approximating systems. So, the problem of finding of coordinates $y=Q(x)$ mentioned in (ii) arises. Such coordinates are called *privileged*; as it is well known, they are not defined uniquely. Different privileged coordinates are constructed, for example, in [1,3,13] and in many other works. In particular, in [5] some explicit form of privileged coordinates is given, in [2] it is shown that privileged coordinates can be chosen polynomially.

In this Note we give necessary and sufficient conditions for coordinates to be privileged (Theorem 3.1 and Corollary 3.2). In particular, our criterion gives an 'algebraic' explanation of the fact that privileged coordinates, in general, do not give a satisfactory information about the precise form of the approximating system $\{\widehat{X}\}$ (analogous to that of [13]). We use an algebraic approach originally proposed in [4], and apply results of [9,11,13]. Our technique is, in a certain sense, close to [7] and [8]. This technique proved to be fruitful for approximations of nonlinear time-optimal control problems [12,13].

In this Note we do not discuss the connection between the definition of a homogeneous approximation adopted here and other definitions introduced before because of the page limitation. We plan to do this in one of our forthcoming papers.

2. An algebraic description of a homogeneous approximation

As it was shown in [4], integral functionals $\eta_{i_1\ldots i_k}=\eta_{i_1\ldots i_k}(\theta,u)$ of the form (3) are linearly independent when (θ,u) runs through the set $\{(\theta,u):\ 0\leqslant\theta\leqslant T,\ \|u\|_\infty\leqslant 1\}$. Hence, a linear span of the mentioned functionals forms *a free associative algebra* over $\mathbb{R}$ with the free generating elements $\{\eta_i:\ 0\leqslant i\leqslant h\}$ and the algebraic operation of 'concatenation'; we denote it by $\vee$. Thus, $\eta_{i_1\ldots i_k}\vee\eta_{j_1\ldots j_s}=\eta_{i_1\ldots i_k j_1\ldots j_s}$, and the set $\{\eta_{i_1\ldots i_k}:\ k\geqslant 1,\ 0\leqslant i_1,\ldots,i_k\leqslant h\}$ is a basis of $\mathcal{F}$ considered as a linear space. In [14] we called this algebra by $\mathcal{F}$ in honor of M. Fliess. Below we keep the notation $a=\sum\alpha_{i_1\ldots i_k}\eta_{i_1\ldots i_k}$ (without arguments) when we mean an element of the algebra $\mathcal{F}$ of functionals and write $a(\theta,u)=\sum\alpha_{i_1\ldots i_k}\eta_{i_1\ldots i_k}(\theta,u)$ for the value of the functional when θ and $u=u(t)$ are given.

Let us introduce a linear operator $c:\mathcal{F}\to\mathbb{R}^n$ defined on basis elements as $c(\eta_{i_1\ldots i_k})=X_{i_k}\cdots X_{i_1}(0)$. Then, the study of many important properties of system (1) can be reduced to the study of 'algebraic properties' of the formal series:

$$\mathcal{E}_X=\sum c(\eta_{i_1\ldots i_k})\eta_{i_1\ldots i_k}.$$

In particular, suppose that an analytic change of variables $y=Q(x)$ reduces the system $\{X\}$ to the system $\{\overline{X}\}$. Since $y_u(t)=Q(x_u(t))$ then $\mathcal{E}_{\overline{X}}(\theta,u)=Q(\mathcal{E}_X(\theta,u))$. Below we show that $\mathcal{E}_{\overline{X}}=Q(\mathcal{E}_X)$. Thus, changes of variables in a system are equivalent to transformations of its series.

The homogeneity property (i) given above is based on the equality $\eta_{i_1\ldots i_k}(\theta, u(\frac{t}{\theta})) = \theta^k \eta_{i_1\ldots i_k}(1, u(t))$. If B_θ denotes the unit ball of the space $L_\infty([0,\theta];\mathbb{R}^h)$ then we get $\eta_{i_1\ldots i_k}(\theta, B_\theta) = \theta^k \eta_{i_1\ldots i_k}(1, B_1)$. Thus, the asymptotic behavior of the integral functional $\eta_{i_1\ldots i_k}(\theta, u)$ as $\theta \to 0$ is described by its "length" k. In algebraic terms, this leads to the *graded structure* in the algebra $\mathcal{F}$, namely, $\mathcal{F} = \sum_{k=1}^{\infty} \mathcal{F}_k$, where $\mathcal{F}_k = \mathrm{Lin}\{\eta_{i_1\ldots i_k}\colon 0 \leqslant i_1, \ldots, i_k \leqslant h\}$. We say that $a \in \mathcal{F}$ is *homogeneous* if $a \in \mathcal{F}_m$ for some $m \geqslant 1$.

Denote by $\mathcal{L}$ the free Lie algebra generated by the set $\{\eta_i\colon 0 \leqslant i \leqslant h\}$ with the Lie brackets $[\ell_1, \ell_2] = \ell_1 \vee \ell_2 - \ell_2 \vee \ell_1$; then $\mathcal{F}$ is the enveloping associative algebra for $\mathcal{L}$. The graded structure in $\mathcal{F}$ induces the graded structure in $\mathcal{L}$, i.e. $\mathcal{L} = \sum_{k=1}^{\infty} \mathcal{L}_k$ where $\mathcal{L}_k = \mathcal{L} \cap \mathcal{F}_k$. System (1) is accessible from the origin iff the Lie algebra rank condition holds, i.e. $c(\mathcal{L}) = \mathbb{R}^n$ [16]. For an approximating system $\{\widehat{X}\}$ we also require $\hat{c}(\mathcal{L}) = \mathbb{R}^n$ where $\hat{c}(\eta_{i_1\ldots i_k}) = \widehat{X}_{i_k} \cdots \widehat{X}_{i_1}(0)$. Thus, properties (i), (ii) can be expressed algebraically as:

(i′) $\hat{c}(\mathcal{L}) = \mathbb{R}^n$ and there exist $1 \leqslant \gamma_1 \leqslant \cdots \leqslant \gamma_n$ such that $(\mathcal{E}_{\widehat{X}})_q \in \mathcal{F}_{\gamma_q}$, $q = 1, \ldots, n$;
(ii′) after a certain transformation $y = Q(x)$, we have $(Q(\mathcal{E}_X) - \mathcal{E}_{\widehat{X}})_q \in \sum_{\gamma > \gamma_q} \mathcal{F}_\gamma$, $q = 1, \ldots, n$.

Here and after $(v)_q$ denotes the q-th component of the vector v, and $(\mathcal{V})_q$ denotes the q-th component $\sum (v_{i_1\ldots i_k})_q \eta_{i_1\ldots i_k}$ of the series $\mathcal{V} = \sum v_{i_1\ldots i_k} \eta_{i_1\ldots i_k}$.

Thus, properties (i′) and (ii′) mean that $\mathcal{E}_{\widehat{X}}$ consists of *leading homogeneous* terms of the series $Q(\mathcal{E}_X)$.

For any $\gamma \in \mathbb{N}$, we consider the subspace:

$$\mathcal{P}_\gamma = \left\{ \ell \in \mathcal{L}_\gamma\colon c(\ell) \in c\left(\sum_{k=1}^{\gamma-1} \mathcal{L}_k\right)\right\}. \tag{4}$$

As it can be shown analogously to [13], the set of these subspaces determines a homogeneous approximation for system (1). Namely, let us construct the *left ideal* $J_X = \sum_{\gamma=1}^{\infty} (\mathcal{F} + \mathbb{R}) \vee \mathcal{P}_\gamma$. Notice that J_X does not depend on the particular choice of coordinates in system (1). Due to the definition, if $a \in J_X \cap \mathcal{F}_\gamma$ then $c(a) \in c(\sum_{k=1}^{\gamma-1} \mathcal{F}_k)$. Hence, elements of J_X *cannot* be considered as leading ones in the series $\mathcal{E}_X$.

Now let us fix an *arbitrary* set of homogeneous elements $\ell_1, \ldots, \ell_n \in \mathcal{L}$ such that $\mathcal{L} = \mathrm{Lin}\{\ell_1, \ldots, \ell_n\} + \sum_{\gamma=1}^{\infty} \mathcal{P}_\gamma$ where $\ell_i \in \mathcal{L}_{\gamma_i}$, $i = 1, \ldots, n$, and $\gamma_1 \leqslant \cdots \leqslant \gamma_n$. This can be done successively: on the i-th step we choose the minimal possible γ_i and $\ell_i \in \mathcal{L}_{\gamma_i}$ such that $c(\ell_i)$ is linearly independent of $c(\ell_1), \ldots, c(\ell_{i-1})$. The required set exists since $c(\mathcal{L}) = \mathbb{R}^n$. As it can be proved analogously to [13,15], the approximating system $\{\widehat{X}\}$ can be chosen so that $(\mathcal{E}_{\widehat{X}})_q = \tilde{\ell}_q$, $q = 1, \ldots, n$, where $\tilde{\ell}_q$ is the orthogonal projection of ℓ_q on the subspace $J_X^\perp$. Emphasize that the elements $\tilde{\ell}_1, \ldots, \tilde{\ell}_n$ are defined *uniquely* up to linear transformations.

3. Lie basis, dual sequences, and an algebraic description of privileged coordinates

As it is well known, elements of a Lie algebra can be used to construct a basis in the enveloping associative algebra. In our case, let $\{\ell_j\}_{j=n+1}^{\infty}$ be a basis of $\sum_{\gamma=1}^{\infty} \mathcal{P}_\gamma$ (consisting of homogeneous elements), i.e. $\{\ell_1, \ldots, \ell_n\} \cup \{\ell_j\}_{j=n+1}^{\infty}$ is a basis of $\mathcal{L}$. Then, by the Poincaré–Birkhoff–Witt theorem, the set $B_{\mathrm{PBW}} = \{\ell_{j_1} \vee \cdots \vee \ell_{j_s}\colon s \geqslant 1,\ j_1 \leqslant \cdots \leqslant j_s\}$ is a basis of $\mathcal{F}$. Another basis is given by the theorem of R. Ree [11]. To formulate it, let us introduce the shuffle product operation $\sqcup\!\sqcup$ in $\mathcal{F}$, defined by the rule:

$$\eta_{i_1\ldots i_k} \sqcup\!\sqcup \eta_{j_1\ldots j_s} = \eta_{i_1} \vee (\eta_{i_2\ldots i_k} \sqcup\!\sqcup \eta_{j_1\ldots j_s}) + \eta_{j_1} \vee (\eta_{i_1\ldots i_k} \sqcup\!\sqcup \eta_{j_2\ldots j_s}), \quad 0 \leqslant i_1, \ldots, i_k, j_1, \ldots, j_s \leqslant h \tag{5}$$

(we assume $\eta_{m_2\ldots m_r} = 1$ if $r = 1$ and $1 \sqcup\!\sqcup a = a \sqcup\!\sqcup 1 = a$ for any $a \in \mathcal{F}$). Then, the Ree's theorem [11] implies that the set $B_R = \{\ell_{j_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \ell_{j_s}\colon s \geqslant 1,\ j_1 \leqslant \cdots \leqslant j_s\}$ is a basis of $\mathcal{F}$. Moreover, this basis has the following remarkable property. Let us introduce the inner product $\langle \cdot, \cdot \rangle$ in $\mathcal{F}$ assuming the basis $\{\eta_{i_1\ldots i_k}\colon k \geqslant 1, 0 \leqslant i_1, \ldots, i_k \leqslant h\}$ is orthonormal (i.e. put $\langle \eta_{i_1\ldots i_k}, \eta_{j_1\ldots j_s} \rangle = 1$ if $k = s$ and $i_r = j_r$, $r = 1, \ldots, k$, and 0 otherwise). Then, the complement subspaces $\mathcal{L} = \mathrm{Lin}\{\ell_j\}_{j=1}^{\infty}$ and $\mathcal{L}^{sh} = \mathrm{Lin}\{\ell_{j_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \ell_{j_s}\colon s \geqslant 2,\ j_1 \leqslant \cdots \leqslant j_s\}$ are *orthogonal* to each other [11]. The generalization of the Ree's theorem with applications to an approximation of nonlinear time-optimal control problems is given in [13].

In 1989 Melançon and Reutenauer [9] found the close connection between the Poincaré–Birkhoff–Witt basis and the shuffle product operation. Let us denote $\ell^p = \ell \vee \cdots \vee \ell$ (p times); then the Poincaré–Birkhoff–Witt basis can be rewritten as $B_{\mathrm{PBW}} = \{\ell_{j_1}^{p_1} \vee \cdots \vee \ell_{j_s}^{p_s}\colon s \geqslant 1, j_1 < \cdots < j_s, p_1, \ldots, p_s \in \mathbb{N}\}$. Suppose $D = \{d_{i_1\ldots i_r}^{q_1\ldots q_r}\colon r \geqslant 1,$

$i_1 < \cdots < i_r, q_1, \ldots, q_r \in \mathbb{N}\}$ is the *dual basis* for B_{PBW}, i.e. $\langle \ell_{j_1}^{p_1} \vee \cdots \vee \ell_{j_s}^{p_s}, d_{i_1\ldots i_r}^{q_1\cdots q_r} \rangle = 1$ if $s = r$, $j_k = i_k$, $p_k = q_k$, $k = 1, \ldots, s$, and $= 0$ otherwise. In our case, it obviously exists since the subspaces $\mathcal{F}_k$ are finite-dimensional and orthogonal to each other. Then, by the Melançon–Reutenauer theorem, $d_{i_1\ldots i_r}^{q_1\cdots q_r} = \frac{1}{q_1!\cdots q_r!} d_{i_1}^{\sqcup\!\sqcup q_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup d_{i_r}^{\sqcup\!\sqcup q_r}$ where $d_i = d_i^1$ and $d^{\sqcup\!\sqcup q} = d \sqcup\!\sqcup \cdots \sqcup\!\sqcup d$ (q times) [9,10].

Thus, let us re-expand the series $\mathcal{E}_X$ w.r.t. the basis D. Due to the duality of D and B_{PBW} we get:

$$\mathcal{E}_X = \sum_{\substack{r\geqslant 1, q_1,\ldots,q_r\in\mathbb{N}\\ 1\leqslant i_1<\cdots<i_r}} \frac{1}{q_1!\cdots q_r!} c\big(\ell_{i_1}^{q_1} \vee \cdots \vee \ell_{i_r}^{q_r}\big) d_{i_1}^{\sqcup\!\sqcup q_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup d_{i_r}^{\sqcup\!\sqcup q_r} = \mathcal{S} + \mathcal{T}, \tag{6}$$

where

$$\mathcal{S} = \sum_{1\leqslant i_1<\cdots<i_r\leqslant n} \frac{1}{q_1!\cdots q_r!} c\big(\ell_{i_1}^{q_1} \vee \cdots \vee \ell_{i_r}^{q_r}\big) d_{i_1}^{\sqcup\!\sqcup q_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup d_{i_r}^{\sqcup\!\sqcup q_r}$$

and

$$\mathcal{T} = \sum_{\substack{1\leqslant i_1<\cdots<i_r\\ i_r\geqslant n+1}} \frac{1}{q_1!\cdots q_r!} c\big(\ell_{i_1}^{q_1} \vee \cdots \vee \ell_{i_r}^{q_r}\big) d_{i_1}^{\sqcup\!\sqcup q_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup d_{i_r}^{\sqcup\!\sqcup q_r}.$$

Notice that each component of $\mathcal{T}$ is a sum of elements with coefficients from $c(J_X)$. Notice also that $d_1, \ldots, d_n \in J_X^{\perp}$ and, therefore, $d_{i_1}^{\sqcup\!\sqcup q_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup d_{i_r}^{\sqcup\!\sqcup q_r} \in J_X^{\perp}$ if $1 \leqslant i_1 < \cdots < i_r \leqslant n$ [13]. Moreover, the set $\{d_{i_1}^{\sqcup\!\sqcup q_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup d_{i_r}^{\sqcup\!\sqcup q_r}\colon r \geqslant 1,\ 1 \leqslant i_1 < \cdots < i_r \leqslant n,\ q_1, \ldots, q_r \in \mathbb{N}\}$ is a basis of $J_X^{\perp}$.

We emphasize that the shuffle product operation corresponds to the pointwise product of iterated integrals, i.e. $(\eta_{i_1\ldots i_k} \sqcup\!\sqcup \eta_{j_1\ldots j_m})(\theta, u) = \eta_{i_1\ldots i_k}(\theta, u) \cdot \eta_{j_1\ldots j_m}(\theta, u)$. Taking this into account, let us apply an analytic nonsingular change of variables $y = Q(x)$ (i.e. such that $Q(0) = 0$ and $\det Q'(0) \neq 0$) to system (1) and consider the obtained system $\{\overline{X}\}$. Since $\mathcal{E}_{\overline{X}}(\theta, u) = Q(\mathcal{E}_X(\theta, u)) = \sum \frac{1}{r!} Q^{(r)}(0)(\mathcal{E}_X(\theta, u))^r$ then

$$\begin{aligned}
\mathcal{E}_{\overline{X}} = \sum \frac{1}{r!} Q^{(r)}(0)(\mathcal{E}_X)^{\sqcup\!\sqcup r} &= \sum \frac{1}{r!} Q^{(r)}(0)(\mathcal{S} + \mathcal{T})^{\sqcup\!\sqcup r} \\
&= \sum \frac{1}{r!} Q^{(r)}(0)(\mathcal{S})^{\sqcup\!\sqcup r} + \sum_{j\geqslant 1} \frac{1}{p!j!} Q^{(p+j)}(0)(\mathcal{S})^{\sqcup\!\sqcup p} \sqcup\!\sqcup (\mathcal{T})^{\sqcup\!\sqcup j} \\
&= Q(\mathcal{S}) + \sum_{j\geqslant 1} \frac{1}{p!j!} Q^{(p+j)}(0)(\mathcal{S})^{\sqcup\!\sqcup p} \sqcup\!\sqcup (\mathcal{T})^{\sqcup\!\sqcup j}.
\end{aligned}$$

Here for a series of the form $\mathcal{V} = \sum v_{i_1\ldots i_k} \eta_{i_1\ldots i_k}$ with components $\mathcal{V}_q = \sum (v_{i_1\ldots i_k})_q \eta_{i_1\ldots i_k}$, we put

$$Q^{(r)}(0)(\mathcal{V})^{\sqcup\!\sqcup r} = \sum_{r_1+\cdots+r_n=r} \frac{r!}{r_1!\cdots r_n!} \frac{\partial^r Q(x)}{\partial x_1^{r_1}\cdots\partial x_n^{r_n}}\bigg|_{x=0} \mathcal{V}_1^{\sqcup\!\sqcup r_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \mathcal{V}_n^{\sqcup\!\sqcup r_n}$$

where the shuffle product is naturally extended to formal series of iterated integrals with real coefficients, and it is assumed that $\mathcal{V}_i^{\sqcup\!\sqcup 0} = 1$. Thus,

$$\mathcal{E}_{\overline{X}} = Q(\mathcal{E}_X) = Q(\mathcal{S}) + \widetilde{\mathcal{T}} \quad \text{where } \widetilde{\mathcal{T}} = \sum_{p\geqslant 0,\ j\geqslant 1} \frac{1}{p!j!} Q^{(p+j)}(0)(\mathcal{S})^{\sqcup\!\sqcup p} \sqcup\!\sqcup (\mathcal{T})^{\sqcup\!\sqcup j}.$$

Each component of $\widetilde{\mathcal{T}}$ is a sum of elements with coefficients from $\bar{c}(J_X)$. Hence, the triangularity of $Q(\mathcal{S})$ guarantees the triangularity of $\mathcal{E}_{\overline{X}} = Q(\mathcal{E}_X)$. Now, we reduce the triangularity property of the 'algebraic object' $Q(\mathcal{S})$ to the triangularity of a vector function. Namely, let us consider:

$$\Phi(z) = \sum_{q_1,\ldots,q_n\geqslant 0} \frac{1}{q_1!\cdots q_n!} c\big(\ell_1^{q_1} \vee \cdots \vee \ell_n^{q_n}\big) z_1^{q_1}\cdots z_n^{q_n}, \quad z \in \mathbb{R}^n \tag{7}$$

(we put $\ell_i^0 = 1$, $1 \vee a = a \vee 1 = a$ for any $a \in \mathcal{F} + \mathbb{R}$, and $c(1) = 0$). One can show that $\Phi(z)$ is analytic and locally invertible in a neighborhood of the origin. Suppose the coordinates $y = Q(x)$ are such that

$$\big(Q\big(\Phi(z)\big)\big)_q = \sum_{\gamma_1 r_1+\cdots+\gamma_n r_n \geqslant \gamma_q} \alpha_q^{r_1,\ldots,r_n} z_1^{r_1}\cdots z_n^{r_n}, \quad q=1,\ldots,n. \tag{8}$$

This means that $(\mathcal{E}_{\overline{X}})_q = \sum_{\gamma_1 r_1+\cdots+\gamma_n r_n=\gamma_q} \alpha_q^{r_1,\ldots,r_n} d_1^{\sqcup\!\sqcup r_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup d_n^{\sqcup\!\sqcup r_n} + \rho_q$, where $\rho_q \in \sum_{\gamma>\gamma_q} \mathcal{F}_\gamma$. Observe that $\alpha_q^{r_1,\ldots,r_n} = \frac{1}{r_1!\cdots r_n!}(\bar{c}(\ell_1^{r_1} \vee \cdots \vee \ell_n^{r_n}))_q$ where $\bar{c}:\mathcal{F}\to\mathbb{R}^n$ is defined as $\bar{c}(\eta_{i_1\ldots i_k}) = \overline{X}_{i_k}\cdots\overline{X}_{i_1}(0)$. If $Q(x)$ is nonsingular then we can choose a system $\{\widehat{X}\}$, accessible from the origin, so that

$$(\mathcal{E}_{\widehat{X}})_q = \sum_{\gamma_1 r_1+\cdots+\gamma_n r_n=\gamma_q} \alpha_q^{r_1,\ldots,r_n} d_1^{\sqcup\!\sqcup r_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup d_n^{\sqcup\!\sqcup r_n}$$

[6,14]. Thus, the following theorem holds:

Theorem 3.1. *The analytic nonsingular mapping $y = Q(x)$ defines privileged coordinates if and only if it reduces mapping* (7) *to the 'triangular' form* (8).

In particular, the coordinates $y = \Phi^{-1}(x)$ are privileged [5]. On the other hand, it is easy to see that instead of $\Phi(z)$ it is sufficient to consider the mapping:

$$\widetilde{\Phi}(z) = \sum_{q_1\gamma_1+\cdots+q_n\gamma_n \leqslant \gamma_n} \frac{1}{q_1!\cdots q_n!} c\big(\ell_1^{q_1} \vee \cdots \vee \ell_n^{q_n}\big) z_1^{q_1}\cdots z_n^{q_n}. \tag{9}$$

Corollary 3.2. *The analytic nonsingular mapping $y = Q(x)$ defines privileged coordinates if and only if it reduces mapping* (9) *to the 'triangular' form* (8).

Since $\widetilde{\Phi}(z)$ is polynomial then privileged coordinates can be chosen in the polynomial form [2,13].

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