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Algebraic Geometry

Smooth toric G-Hilbert schemes via G-graphs

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Abstract

We provide here an infinite family of finite subgroups $\{G_n \subset \mathrm{SL}_n(\mathbb{C})\}_{n \geqslant 2}$ for which the G-Hilbert scheme $G_n\text{-}\mathrm{Hilb}\,\mathbb{A}^n$ is a crepant resolution of $\mathbb{A}^n/G_n$, via the Hilbert–Chow morphism. The proof is based on an explicit description of the toric structure of $G_n\text{-}\mathrm{Hilb}\,\mathbb{A}^n$ in terms of Nakamura's G_n-graphs. ***To cite this article: M. Sebestean, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

G*-schémas de Hilbert toriques lisses à l'aide des *G*-graphes.** Nous décrivons ici une famille infinie de sous-groupes finis $\{G_n \subset \mathrm{SL}_n(\mathbb{C})\}_{n \geqslant 2}$, telle que le G_n-schéma de Hilbert sur l'espace affine $\mathbb{A}^n$ soit lisse et donne une résolution crépante de $\mathbb{A}^n/G_n$, pour tout $n \geqslant 2$, via le morphisme de Hilbert–Chow. La preuve est basée sur une description explicite de la structure torique de $G_n\text{-}\mathrm{Hilb}\,\mathbb{A}^n$, $n \geqslant 2$, à l'aide de G_n-graphes. ***Pour citer cet article : M. Sebestean, C. R. Acad. Sci. Paris, Ser. I 344 (2007).

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Soit $n \geqslant 2$ un entier positif et G un sous-groupe abélien fini de $\mathrm{SL}_n(\mathbb{C})$. Si le groupe G ne contient pas de pseudoréflections, le quotient $\mathbb{A}^n/G$ est une singularité canonique de Gorenstein [12]. Ito et Nakamura [6] ont introduit la notion de G-schéma de Hilbert et demandé quand celui-ci est une résolution crépante du quotient $\mathbb{A}^n/G$. Cette question a une réponse positive pour $n \leqslant 3$ [1,2,5,8]. En dimension $n \geqslant 4$, la question a rarement une réponse positive. La variété $G\text{-}\mathrm{Hilb}\,\mathbb{A}^n$ s'avère être en effet très singulière et le morphisme de Hilbert–Chow n'est pas en général crépant.

Nakamura [8] décrit explicitement le schéma $G\text{-}\mathrm{Hilb}\,\mathbb{A}^n$, dans le cas où G est un sous-groupe abélien fini de $\mathrm{SL}_n(\mathbb{C})$, $n \geqslant 2$, en utilisant la notion de G-graphe. Il montre en effet que le G-schéma de Hilbert de $\mathbb{A}^n$ est la variété obtenue en recollant les $\mathrm{Spec}[\mathbb{C}(S(\Gamma)]$, où Γ parcourt l'ensemble de tous les G-graphes. Ce résultat ne fournit cependant aucune information sur la lissité de $G\text{-}\mathrm{Hilb}\,\mathbb{A}^n$ pour $n \geqslant 4$.

Le but de cette Note est de décrire une famille infinie $\{G_n\}_{n \geqslant 2}$ de sous-groupes finis de $\mathrm{SL}_n(\mathbb{C})$ pour lesquels $G_n\text{-}\mathrm{Hilb}\,\mathbb{A}^n$ est une résolution crépante de $\mathbb{A}^n/G_n$. Pour un entier positif $n \geqslant 2$ fixé, on considère $\varepsilon = \exp(\frac{2\pi \mathrm{i}}{2^n-1})$ une racine primitive d'ordre $2^n - 1$ de l'unité. Soit G_n le sous-groupe cyclique de $\mathrm{SL}_n(\mathbb{C})$ engendré par la matrice

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diagonale $\mathrm{diag}(\varepsilon, \varepsilon^2, \varepsilon^{2^2}, \dots, \varepsilon^{2^{n-1}})$. Pour un G_n-graphe Γ, on note $pv(\Gamma)$ le vecteur dont la i-ème coordonnée est $mp_\Gamma(X_i) := \max\{k \in \mathbb{N} \mid X_i^k \in \Gamma\}$, où les X_i sont les variables de l'anneau des polynômes $\mathbb{C}[X_1, \dots, X_n]$.

Lemme 0.1. *Un G_n-graphe Γ est un ensemble de $2^n - 1$ monômes tel que $pv(\Gamma)$ est égal à :*

(i) (G_n*-graphes linéaires*) $(\underbrace{0, \dots, 0}_{b \text{ fois}}, 2^n - 2, \underbrace{0, \dots, 0}_{n-b-1 \text{ fois}})$, *pour b entier positif compris entre* 0 *et* $n - 1$.

(ii) (G_n*-graphes non-linéaires*) $(\underbrace{0, \dots, 0}_{b \text{ fois}}, 2^{i_1+1} - 1, \underbrace{0, \dots, 0}_{i_1 \text{ fois}}, 2^{i_2+1} - 1, \underbrace{0, \dots, 0}_{i_2 \text{ fois}}, \dots, 2^{i_{n-k}+1} - 1, \underbrace{0, \dots, 0}_{i_{n-k}-b \text{ fois}})$, *pour* $1 \leqslant k \leqslant n - 2$, $0 \leqslant i_j \leqslant k$ *tels que* $i_1 + \dots + i_{n-k} = k$ *et* $0 \leqslant b \leqslant i_{n-k}$. *Pour un tel G_n-graphe, le monôme* $\prod_{i=1}^n X_i^{mp_\Gamma(X_i)}$ *n'appartient pas à* Γ.

La démonstration de ce résultat est basée sur des méthodes combinatoires et l'application de ce lemme s'avère être le point-clé dans la preuve de la proposition suivante :

Proposition 0.2. *Soit Γ un G_n-graphe tel que $pv(\Gamma) = (c_1, \dots, c_n)$.*

(i) *Le cône $\sigma(\Gamma)$ est engendré par* :
   (a) *les vecteurs e_i de la base canonique, pour les indices i tels que $c_i \neq 0$,*
   (b) *les permutations cycliques à gauche de $n - i + 1$ crans du vecteur* $g_n := \frac{1}{2^n-1}(1, 2, 4, \dots, 2^n - 1)$, *pour les indices i tels que $c_i = 0$.*

(ii) *Si N désigne le réseau $\mathbb{Z}^n + g_n\mathbb{Z}$, alors $S(\Gamma)$ est égal à $\sigma^\vee(\Gamma) \cap N^\vee$, où $\sigma^\vee(\Gamma)$ (resp. $N^\vee$) désigne le dual de $\sigma(\Gamma)$ (resp. N).*

La proposition précédente permet alors de conclure :

Théorème 0.3. *Pour tout entier positif $n \geqslant 2$, le G_n-schéma de Hilbert de $\mathbb{A}^n$ est une variété lisse, résolution crépante du quotient $\mathbb{A}^n/G_n$, via le morphisme de Hilbert–Chow.*

1. Introduction

Let $n \geqslant 2$ and $G \subset \mathrm{SL}_n(\mathbb{C})$ a finite subgroup. The orbit space $\mathbb{A}^n/G$ is a canonical Gorenstein singularity [12]. Ito and Nakamura [6] introduced the notion of G-Hilbert scheme and asked when it is a crepant resolution of $\mathbb{A}^n/G$. Different authors [1], [2,5] show that this holds for $n = 3$ and that, moreover, the Hilbert–Chow morphism $\pi : G\text{-}\mathrm{Hilb}\,\mathbb{A}^3 \to \mathbb{A}^3/G$ is crepant. For $n \geqslant 4$, the conjecture is false in general: the variety $G\text{-}\mathrm{Hilb}\,\mathbb{A}^n$ is singular and the Hilbert–Chow morphism is discrepant.

For a finite diagonal Abelian subgroup G of $\mathrm{SL}_n(\mathbb{C})$ with $n \geqslant 2$, Nakamura [8] describes the G-Hilbert scheme of $\mathbb{A}^n$ explicitly in terms of G-graphs. We consider the case of a cyclic group G. Let r denote the order of G and fix ε a primitive root of unity of order r. We write an element g of G as a diagonal matrix $\mathrm{diag}(\varepsilon^{w_{1,g}}, \dots, \varepsilon^{w_{n,g}})$, where $w_{i,g}$ for $1 \leqslant i \leqslant n$ are integers between 0 and $r - 1$.

The group G acts on the affine space $\mathbb{A}^n$, hence on the polynomial ring $\mathbb{C}[X_1, \dots, X_n]$. It is enough to consider its action on the lattice M of all Laurent monomials in n variables. As above, we say that g acts by weight $w_{i,g}$ on the variable X_i. Let $p = X_1^{p_1} \cdots X_n^{p_n}$ be a Laurent monomial. The residue modulo r of the sum $\sum_{i=1}^n w_{i,g} p_i$ is denoted by $\mathrm{wt}_g(p)$ and is called the weight of p with respect to g. Thus, the action of G on M is given by $g \cdot p = \varepsilon^{\mathrm{wt}_g(p)} p$, for all $p \in M$ and all $g \in G$. For χ an irreducible character of G and p a Laurent monomial, we say that χ and p are associated if $g \cdot p = \chi(g) p$ for any g of G or, equivalently, if $\chi(g) = \varepsilon^{\mathrm{wt}_g(p)}$ holds for any g. We can now introduce the following definition (cf. [8], Definition 1.4).

Definition 1.1. A subset Γ of monomials is called a G-graph if:

(i) it contains the constant monomial 1;

(ii) if p is a monomial of Γ and the monomial q divides p, then q is also in Γ;
(iii) the map sending a monomial to its associated character is a bijection from Γ to the set of all irreducible characters of G.

We denote by Graph G the set of all G-graphs. Condition (iii) says that for a given monomial p of M, there exists a unique monomial $\mathrm{wt}_\Gamma(p)$ of Γ with the same associated character. Let v denote the natural bijection between M and $\mathbb{Z}^n$. For each G-graph Γ, define a cone $\sigma(\Gamma) := \{u \in \mathbb{R}^n\colon \langle u, v(p/\mathrm{wt}_\Gamma(p))\rangle \geqslant 0, \forall p \in M\}$, and denote by Fan G the set of all cones $\sigma(\Gamma)$ for $\Gamma \in$ Graph G. For a G-graph Γ, let $S(\Gamma)$ be the sub-semigroup of $N^\vee$ generated by the vectors $v(p/\mathrm{wt}_\Gamma(p))$, $p \in M$, and denote by $V(\Gamma)$ the variety $\mathrm{Spec}\,\mathbb{C}[S(\Gamma)]$.

Nakamura [8] proves that the scheme obtained by gluing the $V(\Gamma)$ for $\Gamma \in$ Graph G is the G-Hilbert scheme of $\mathbb{A}^n$. For a general group G, this result does not give much information concerning the smoothness of G-Hilb $\mathbb{A}^n$. The aim of this paper is to provide an infinite family of linear groups $\{G_n\}_{n\geqslant 2}$ for which the corresponding G-Hilbert schemes of $\mathbb{A}^n$ are smooth and the Hilbert–Chow morphism is crepant. The proof of this result is based on an explicit description of the G_n-graphs.

Proofs are omitted or only sketched (full proofs are given in [10] and will appear elsewhere).

2. G_n-graphs

Given an integer $n \geqslant 2$, we write $r = 2^n - 1$ and fix the rth primitive root of unity $\varepsilon = \exp(\frac{2\pi \mathrm{i}}{r})$. Let G_n be the cyclic subgroup of $\mathrm{SL}_n(\mathbb{C})$ generated by the diagonal matrix $g_n := \mathrm{diag}(\varepsilon, \varepsilon^2, \varepsilon^{2^2}, \ldots, \varepsilon^{2^{n-1}})$. Let Γ be a G_n-graph. We denote by $pv(\Gamma)$ the power vector of Γ defined as the vector whose ith coordinate is $mp_\Gamma(X_i) := \max\{k \in \mathbb{N} \mid X_i^k \in \Gamma\}$.

Lemma 2.1. *A G_n-graph Γ is a set of $2^n - 1$ monomials whose power vector $pv(\Gamma)$ equals*

(i) (*linear G_n-graphs*) $(\underbrace{0, \ldots, 0}_{b \text{ times}}, 2^n - 2, \underbrace{0, \ldots, 0}_{n-b-1 \text{ times}})$ *for b an integer between* 0 *and* $n - 1$.
(ii) (*nonlinear G_n-graphs*) $(\underbrace{0, \ldots, 0}_{b}, 2^{i_1+1} - 1, \underbrace{0, \ldots, 0}_{i_1}, 2^{i_2+1} - 1, \underbrace{0, \ldots, 0}_{i_2}, \ldots, 2^{i_{n-k}+1} - 1, \underbrace{0, \ldots, 0}_{i_{n-k}-b})$ *for* $1 \leqslant k \leqslant n - 2$, $0 \leqslant i_j \leqslant k$ *such that* $i_1 + \cdots + i_{n-k} = k$ *and* $0 \leqslant b \leqslant i_{n-k}$. *For such a G_n-graph, the monomial* $\prod_{j=1}^n X_j^{mp_\Gamma(X_j)}$ *does not belong to* Γ.

Remark 1. Lemma 2.1 describes a G_n-graph Γ directly. If Γ contains only monomials in one variable, it is of the form $\{1, X_j, X_j^2, \ldots, X_j^{2^n-2}\}$. A nonlinear Γ consists of the monomials $\prod_{j=1}^n X_j^{a_j}$ with $0 \leqslant a_j \leqslant mp_\Gamma(X_j)$ such that $\sum_{j=1}^n a_j < \sum_{j=1}^n mp_\Gamma(X_j)$.

2.1. Proof of Lemma 2.1

First, as in the remark, we show that any set Γ of $2^n - 1$ monomials and power vector $pv(\Gamma)$ as in (i) or (ii) is a G_n-graph. We need to prove condition (iii) of Definition 1.1. Because G_n is a cyclic group, it is enough to consider the weight of a monomial $p \in M$ with respect to the generator g_n. We call this the weight of p and denote it by $\mathrm{wt}(p)$. It is the remainder modulo $2^n - 1$ of the weighted sum of the exponents $\sum_{j=1}^n a_j \cdot 2^{j-1}$. To conclude, we compute the weight of each monomial of Γ.

Next, we prove that the power vector of any G_n-graph Γ is given by (i) or (ii). Let k be the number of variables that do not occur in any monomial of Γ. For such a variable, the corresponding mp_Γ is zero. If $k = n - 1$, let X_j, $1 \leqslant j \leqslant n$, be the unique variable that occurs in Γ. An argument on the weights then shows that Γ has to be a linear G_n-graph, as in (i). If $k \leqslant n - 2$, we prove that Γ has the form of (ii). Suppose that the first b variables (possible zero) do not occur in Γ. We denote by i_l, $1 \leqslant i_l \leqslant n - k$, the increasing chain of indices for which $mp_\Gamma(X_{i_j})$ is positive. We have $i_1 = b + 1$. We argue by contradiction and use an argument on weights to show that for all l, $1 \leqslant l \leqslant n - k$, the integer $mp_\Gamma(X_{i_l})$ has the form $2^{i_l} - 1$. Then we deduce that $i_l = b + l + i_1 + i_2 + \cdots + i_{l-1}$, $2 \leqslant l \leqslant n - k$. That implies the stated relations between n, k and the indices i_l, $1 \leqslant l \leqslant n - k$. □

Remark 2. The proof of Lemma 2.1 is based only on combinatorial methods and particular properties of the groups G_n. Compare this with Nakamura's methods ([8], Theorem 2.11(i)), that describe $\operatorname{Graph} G$ for an Abelian group $G \subset \mathrm{SL}_n(\mathbb{C})$. Indeed, one can prove that the set of all G_n-graphs can be recovered from a given G_n-graph. For an explicit description in dimension four, see [11].

The bijection between the set of all monomials in n variables and the lattice $\mathbb{Z}^n$ allows us to compute the cones $\sigma(\Gamma)$ and the semigroups $S(\Gamma)$ associated to a G_n-graph Γ. The key point in this description is Lemma 2.1, which provides the proof of the following proposition:

Proposition 2.2. *Let Γ be a G_n-graph with $pv(\Gamma) = (c_1, \ldots, c_n)$.*

(i) *The cone $\sigma(\Gamma)$ is generated by*:
  (a) *the vectors e_i of the canonical basis for i such that $c_i \neq 0$,*
  (b) *the vector $g_n := \frac{1}{2^n-1}(1, 2, 4, \ldots, 2^n - 1)$, and its cyclic permutations $n - i + 1$ steps to the left, for i such that $c_i = 0$.*
(ii) *If N is the lattice $\mathbb{Z}^n + g_n\mathbb{Z}$, then the semigroup $S(\Gamma)$ equals $\sigma^\vee(\Gamma) \cap N^\vee$, where $\sigma^\vee(\Gamma)$ (resp. $N^\vee$) denotes the dual of $\sigma(\Gamma)$ (resp. N).*

3. Smoothness of the G_n-Hilbert schemes

For any group G_n, $n \geqslant 2$, the quotient $\mathbb{A}^n/G_n$ is a toric variety with an isolated Gorenstein singularity at the origin. Its lattice is the lattice $N = \mathbb{Z}^n + g_n\mathbb{Z}$, $g_n = \frac{1}{2^n-1}(1, 2, 4, \ldots, 2^n - 1)$, and its fan is the cone $\sigma_0 := \langle e_1, \ldots, e_n \rangle$, where $\{e_i\}_{1 \leqslant i \leqslant n}$ is the canonical basis of $\mathbb{C}^n$. We obtain a toric description of the G_n-Hilbert scheme of $\mathbb{A}^n$ as follows:

Theorem 3.1. *The variety obtained by gluing together the affine pieces $\operatorname{Spec} \mathbb{C}[\sigma^\vee(\Gamma) \cap N^\vee]$ for all $\Gamma \in \operatorname{Graph} G_n$ is the G_n-Hilbert scheme of $\mathbb{A}^n$. It is a smooth variety and a crepant resolution of the singularity $\mathbb{A}^n/G_n$ via the Hilbert–Chow morphism $\pi_n : G_n\text{-}\operatorname{Hilb} \mathbb{A}^n \to \mathbb{A}^n/G_n$.*

3.1. Proof

Denote by X_n the variety obtained by gluing the $\operatorname{Spec} \mathbb{C}[\sigma^\vee(\Gamma) \cap N^\vee]$ for $\Gamma \in \operatorname{Graph} G_n$. The equality of Proposition 2.2(ii) and [8], Theorem 2.11(iii) show that X_n is the G_n-Hilbert scheme of $\mathbb{A}^n$. In particular, by (iv) of the cited Theorem, we deduce that $G_n\text{-}\operatorname{Hilb} \mathbb{A}^n$ is normal.

By Proposition 2.2(i), the generators of a cone $\sigma(\Gamma)$, $\Gamma \in \operatorname{Graph} G_n$, are n linearly independent vectors that form a basis of the lattice N. We deduce that any cone $\sigma(\Gamma)$ is a n-dimensional cone and the associated variety $\operatorname{Spec} \mathbb{C}[\sigma^\vee(\Gamma) \cap N^\vee] = V(\Gamma)$ is smooth. Thus X_n is also smooth and the resolution map is the Hilbert–Chow morphism π_n. The subdivision of σ_0 into $\operatorname{Fan} G_n$ is realized by primitive vectors (that is, those with sum of the coordinates one). Thus, general results for toric varieties (see [9]) provide the crepancy of π_n.

Remark 3. Note that the theorem can also be proved by a direct computation on the dimension of the tangent spaces. The G-graph approach is a combinatorial method that also gives a toric description of the G-Hilbert scheme.

Remark 4. The example we provide in this note is not covered by previous results. The groups G_n are not symplectic for $n \geqslant 3$, the toric varieties we deal with are not local complete intersections (so one cannot apply [3,4]) and for $n \geqslant 4$, [1] does not apply because the exceptional locus is too big.

Nevertheless, all good properties one can expect from such a resolution hold. The description of $\operatorname{Fan} G_n$ shows in particular that the Euler number of $G_n\text{-}\operatorname{Hilb} \mathbb{A}^n$ is $2^n - 1$, so that the strong McKay correspondence holds. The derived McKay correspondence is true for the family $\{G_n\}_{n \geqslant 2}$, following a result of Kawamata [7] and computations described in [10].

References

[1] T. Bridgeland, A. King, M. Reid, The McKay correspondence as an equivalence of derived categories, J. Amer. Math. Soc. 14 (3) (2001) 535–554.
[2] A. Craw, M. Reid, How to calculate $A\text{-}\mathrm{Hilb}\,\mathbb{C}^3$, Séminaires et Congrès 6 (2002) 129–154.
[3] D. Dais, C. Haase, G. Ziegler, All toric local complete intersection singularities admit projective crepant resolutions, Tohoku Math. J. (2) 53 (1) (2001) 95–107.
[4] D. Dais, M. Henk, G. Ziegler, All Abelian quotient C.I.-singularities admit projective crepant resolutions in all dimensions, Adv. Math. 139 (2) (1998) 194–239.
[5] Y. Ito, H. Nakajima, McKay correspondence and Hilbert schemes in dimension three, Topology 39 (6) (2000) 1155–1191.
[6] Y. Ito, I. Nakamura, McKay correspondence and Hilbert schemes, Proc. Japan Acad. Ser. A 72A (1996) 135–138.
[7] Y. Kawamata, Log crepant birational maps and derived categories, J. Math. Sci. Univ. Tokyo 12 (2) (2005) 211–231.
[8] I. Nakamura, Hilbert schemes of Abelian group orbits, J. Algebraic Geom. 10 (4) (2001) 757–779.
[9] M. Reid, Young person's guide to canonical singularities, in: Algebraic Geometry, Part 1, Bowdoin, 1985, Brunswick, Maine, 1985, in: Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 345–414.
[10] M. Sebestean, Correspondance de McKay et équivalences dérivées, PhD thesis, Paris 7 "Denis Diderot" University.
[11] M. Sebestean, A smooth four-dimensional G-Hilbert scheme, Serdica Math. J. 30 (2–3) (2004) 283–292.
[12] K. Watanabe, Certain invariant subrings are Gorenstein, I, II, Osaka J. Math. 11 (1974) 1–8, Osaka J. Math. 11 (1974) 379–388.