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C. R. Acad. Sci. Paris, Ser. I 344 (2007) 327–330

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Probability Theory

Inequalities for characteristic functions involving Fisher information

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Received 20 June 2006; accepted after revision 16 January 2007

Available online 20 February 2007

Presented by Paul Malliavin

Abstract

We establish several inequalities for characteristic functions (Fourier transform of probability densities) in terms of the Fisher information. As applications, we illustrate their significance in estimating the survival probability of a quantum state (Schrödinger wave function). ***To cite this article: Z. Zhang, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Certaines inégalités pour les fonctions caractéristiques faisant intervenir l'information de Fisher. Nous établissons plusieurs inégalités concernant les fonctions caractéristiques (les transformées de Fourier des densités de probabilité) à l'aide de l'information de Fisher. En application, nous montrons la signification des ces inégalités dans l'estimation de la probabilité de survie d'un état quantique (fonction d'onde de Schrödinger). ***Pour citer cet article : Z. Zhang, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

1. Introduction

Let $f(x)$ be a probability density function on the real line R with characteristic function

$$\phi(t)=\int_R \mathrm{e}^{\mathrm{i}tx} f(x)\,\mathrm{d}x,\quad t\in R.$$

In the language of harmonic analysis, $\phi(-t)$ is essentially the Fourier transform of $f(x)$.

It is of interest, both for theoretical reasons and for application purposes, to give bounds for $|\phi(t)|$. A simple yet useful lower bound is

$$\left|\phi(t)\right|\geqslant 1-\frac{1}{2}M_2|t|^2,\quad \forall t\in R,$$

or more generally, $|\phi(t)|\geqslant 1-c_\beta M_\beta|t|^\beta$ for any $0\leqslant\beta\leqslant 2$ and $t\in R$ [7]. Here $M_\beta=\int_R|x|^\beta f(x)\,\mathrm{d}x$ and c_β is a positive constant. Other related bounds are given in [8,9].

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2007.01.008

In this Note, we will establish some *upper* bounds for $|\phi(t)|$ by use of the Fisher information. In particular, we obtain an inequality dual to the above inequality in a loose sense.

Concerning upper bounds for $\phi(t)$, the first coming to mind is, of course, the trivial inequality $|\phi(t)| \leqslant 1$. A less trivial result is that if there exist constants a and b such that $|\phi(t)| \leqslant a < 1$ whenever $|t| \geqslant b$, then $|\phi(t)| < 1 - \frac{1-a^2}{8b^2}t^2$ for all t such that $0 < |t| < b$ ([6], page 61). The disadvantages of this inequality lie in that it is local and the condition may be difficult to verify since itself involves upper bound estimates for $\phi(t)$. On the other hand, we have the following two alternative upper bounds [2,10]: $|\phi(t)|^2 \leqslant \frac{1}{2}(1 + |\phi(2t)|)$, and $|\phi(t)| \leqslant |\phi(nt)|\frac{1}{n}\sin(\frac{\pi}{2n}) + \cos(\frac{\pi}{2n})$, $n = 1, 2, \ldots.$ These inequalities are self-referencing in the sense that they set upper bounds for $|\phi|$ at some point t in terms of the value of $|\phi|$ at some other point. Our main result is in the same spirit as these two estimates, but incorporates one more crucial characteristic, viz., the Fisher information, of the probability density.

Theorem 1. *Let $f(x)$ be a continuously differentiable probability density on R with characteristic function $\phi(t)$, then*

$$\phi_{\Re}^2(t) + \frac{t^2}{I(f)}\phi_{\Im}^2(t) \leqslant \frac{1 + \phi_{\Re}(2t)}{2}, \tag{1}$$

$$\phi_{\Im}^2(t) + \frac{t^2}{I(f)}\phi_{\Re}^2(t) \leqslant \frac{1 - \phi_{\Re}(2t)}{2}. \tag{2}$$

Here $\phi_{\Re}(t)$ and $\phi_{\Im}(t)$ denote the real and the imaginary parts, respectively, of $\phi(t)$ (note that $\phi_{\Re}^2(t)$ is the square of $\phi_{\Re}(t)$, not *the real part of $\phi^2(t)$, etc.), and*

$$I(f) = \int_R \left(\frac{\partial}{\partial x}\ln f(x)\right)^2 f(x)\,\mathrm{d}x \tag{3}$$

is the Fisher information of f, as defined in [5]*, and is assumed to be finite.*

Some simple consequences of Theorem 1 are as follows. Firstly, by adding inequalities (1) and (2), we have

$$\left|\phi(t)\right|^2 \leqslant \frac{I(f)}{I(f) + t^2}, \quad \forall t \in R, \tag{4}$$

which is of particular interest and may be regarded as dual to $|\phi(t)| \geqslant 1 - \frac{1}{2}M_2|t|^2$.

Secondly, by multiplying inequalities (1) and (2), we obtain

$$\frac{t^2}{I(f)}\left|\phi(t)\right|^4 + \left(1 - \frac{t^2}{I(f)}\right)^2 \phi_{\Re}^2(t)\phi_{\Im}^2(t) \leqslant \frac{1 - \phi_{\Re}^2(2t)}{4}, \quad \forall t \in R.$$

In particular, if ϕ is real-valued, then

$$\left|\phi(t)\right|^4 \leqslant \frac{1 - |\phi(2t)|^2}{4t^2} I(f), \quad \forall t \in R.$$

Corollary 1. *Let $M(f) = \int_R |x| f(x)\,\mathrm{d}x$, which is assumed to be finite, then*

$$I(f)M(f) \leqslant \frac{1}{\pi}\int_R \left(\frac{I(f)}{t^2}\left(1 - \phi_{\Re}^2(t)\right) - \phi_{\Im}^2(t)\right)\mathrm{d}t, \tag{5}$$

$$I(f)M(f) \geqslant \frac{1}{\pi}\int_R \left(\phi_{\Re}^2(t) + \frac{I(f)}{t^2}\phi_{\Im}^2(t)\right)\mathrm{d}t. \tag{6}$$

In particular, if ϕ is real-valued, then

$$I(f)M(f) \geqslant \frac{1}{\pi}\int_R \phi^2(t) = 2\int_R f^2(x)\,\mathrm{d}x. \tag{7}$$

The quantity $\int_R f^2(x)\,\mathrm{d}x$ is also called the information energy or the quadratic entropy (and is simply related to Rényi's entropy of order 2). It appears in several important contexts such as quantum mechanics [3], statistical estimation [4], and information theory [11].

The proof will be given in Section 2 and some applications will be presented in Section 3.

2. Proof of the main result

We first prepare a lemma, which is a direct consequence of the Cramér–Rao inequality.

Lemma 1. *Let X be a random variable with density $f(x)$ on R such that $I(f) < \infty$, and let $\xi(x)$ be a real-valued bounded function with bounded continuous derivative, then*

$$\Delta\xi(X) \geqslant \frac{(\mathrm{E}\xi'(X))^2}{I(f)}. \tag{8}$$

Here Δ denotes variance, E *denotes expectation, and the prime $'$ denotes derivative.*

To establish (8), let us put $f_\theta(x) = f(x-\theta)$ and regard $\theta \in R$ as a parameter. Let Δ_θ denote the variance, and E_θ the expectation, with respect to f_θ. Then the standard Cramér–Rao inequality states that [1]

$$\Delta_\theta\xi(X) \geqslant \frac{(\frac{\partial}{\partial\theta}\mathrm{E}_\theta\xi(X))^2}{I(f_\theta)}, \tag{9}$$

where the Fisher information $I(f_\theta)$ with respect to the parameter θ is defined as

$$I(f_\theta) = \int_R \left(\frac{\partial}{\partial\theta}\ln f_\theta(x)\right)^2 f_\theta(x)\,\mathrm{d}x,$$

which turns out to be equal to $I(f)$ defined by Eq. (3) because $f_\theta(x) = f(x-\theta)$. Furthermore,

$$\begin{aligned}\frac{\partial}{\partial\theta}\mathrm{E}_\theta\xi(X) &= \frac{\partial}{\partial\theta}\int_R \xi(x)f(x-\theta)\,\mathrm{d}x = \frac{\partial}{\partial\theta}\int_R \xi(x+\theta)f(x)\,\mathrm{d}x \\ &= \int_R \frac{\partial}{\partial\theta}\xi(x+\theta)f(x)\,\mathrm{d}x = \int_R \xi'(x+\theta)f(x)\,\mathrm{d}x \\ &= \mathrm{E}\xi'(X+\theta).\end{aligned}$$

Consequently, inequality (8) follows from inequality (9) by putting $\theta = 0$.

Now we proceed to the proof of Theorem 1. For any fixed $t \in R$, take $\xi(x) = \cos(tx)$ in the context of Lemma 1, then $\mathrm{E}\xi'(X) = -t\phi_{\Im}(t)$, and

$$\Delta\xi(X) = \mathrm{E}\cos^2(tX) - \big(\mathrm{E}\cos(tX)\big)^2 = \frac{1}{2}\big(1+\phi_{\Re}(2t)\big) - \phi_{\Re}^2(t).$$

Consequently, inequality (1) follows by substituting the above expressions into inequality (8).

Similarly, take $\xi(x) = \sin(tx)$, then $\mathrm{E}\xi'(X) = t\phi_{\Re}(t)$, and

$$\Delta\xi(X) = \mathrm{E}\sin^2(tX) - \big(\mathrm{E}\sin(tX)\big)^2 = \frac{1}{2}\big(1-\phi_{\Re}(2t)\big) - \phi_{\Im}^2(t).$$

Accordingly, inequality (2) also follows by substituting the above quantities into inequality (8).

Inequalities (5) and (6) follow from integrating inequalities (1) and (2), respectively, and taking into account the identity $\int_R \frac{1-\phi_{\Re}(t)}{t^2}\,\mathrm{d}t = \pi\int_R |x|f(x)\,\mathrm{d}x$.

By the Parseval theorem, we have $\int_R |\phi(t)|^2\,\mathrm{d}t = 2\pi\int_R f^2(x)\,\mathrm{d}x$, and consequently, inequality (7) follows from inequality (6).

3. Applications

Let us illustrate some physical implications of the results. Following physicist's terminology and considering the evolution of a quantum state $|\psi\rangle$ driven by a time-independent energy observable H. The survival amplitude at time t is defined as

$$\phi(t) = \langle\psi|\mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle, \quad t \in R,$$

where $\hbar$ is the Planck constant divided by 2π. Now assume that the energy spectrum $\{|x\rangle\}$ of H is continuous, then $|\psi\rangle$ can be expanded in terms of $\{|x\rangle\}$ as $|\psi\rangle = \int_R \lambda(x)|x\rangle\,\mathrm{d}x$ with $\lambda(x) = \langle x|\psi\rangle$, thus $\mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle = \int_R \mathrm{e}^{-\mathrm{i}tx/\hbar}\lambda(x)|x\rangle\,\mathrm{d}x$. By the Parseval theorem,

$$\phi(t) = \langle\psi|\mathrm{e}^{-\mathrm{i}tH/\hbar}|\psi\rangle = \int_R \mathrm{e}^{-\mathrm{i}tx/\hbar}\big|\lambda(x)\big|^2\,\mathrm{d}x.$$

Consequently, the survival amplitude is precisely the characteristic function of the state probability density $|\lambda(x)|^2 = |\langle x|\psi\rangle|^2$ in the energy representation if we replace t by $-t/\hbar$. By inequality (4), we conclude that

$$\big|\phi(t)\big|^2 \leqslant \frac{I(|\lambda|^2)\hbar^2}{I(|\lambda|^2)\hbar^2 + t^2}, \quad t \in R.$$

Thus for large time, the survival probability decays at least as $1/t^2$ if the Fisher information $I(|\lambda|^2)$ is finite. Moreover, if we define the average lifetime $\tau = \int_0^\infty |\phi(t)|^2\,\mathrm{d}t$, then

$$\tau \leqslant \int_0^\infty \frac{I(|\lambda|^2)\hbar^2}{I(|\lambda|^2)\hbar^2 + t^2}\,\mathrm{d}t = \frac{1}{2}\pi\hbar\sqrt{I\big(|\lambda|^2\big)},$$

which sets an upper bound for the average lifetime in terms of the Fisher information. Moreover, if ϕ is real-valued, then by inequality (7), we have

$$\tau \leqslant \frac{1}{2}\pi I\big(|\lambda|^2\big)M\big(|\lambda|^2\big).$$

Note here $M(|\lambda|^2)$ has a simple physical interpretation as the absolute average energy.

Acknowledgements

The author is grateful to the referee for very helpful suggestions.

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