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C. R. Acad. Sci. Paris, Ser. I 344 (2007) 317–320
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Differential Geometry

Elliptic genera of level N on complex π_2-finite manifolds

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Received 27 June 2006; accepted after revision 22 January 2007

Presented by Jean-Pierre Demailly

Abstract

We prove the rigidity of the elliptic genera of level N on complex manifolds with finite second homotopy group admitting circle actions, and the vanishing of the Hilbert polynomial of its canonical bundle. ***To cite this article: R. Herrera, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Genres elliptiques du niveau N sur variétés complexes avec le deuxième groupe homotopie fini. On montre la rigidité des genres elliptiques de niveau N sur les variétés complexes avec deuxième groupe d'homotopie fini et dotées d'actions de S^1, et l'annulation du polynôme de Hilbert de son fibré vectoriel canonique. ***Pour citer cet article : R. Herrera, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

1. Introduction

The elliptic genus was introduced by Ochanine [6] and re-interpreted by Witten [8], who conjectured its rigidity under circle actions on spin manifolds. The rigidity of the elliptic genus was proved by Taubes [7], Bott and Taubes [1], etc., and was generalized to non-spin manifolds with finite second homotopy group in [3]. Furthermore, Witten and Hirzebruch proposed independently a complex version of the genus in the form of the elliptic genus of level $N > 0$ for complex manifolds with $c_1 \equiv 0 \pmod N$ and conjectured its rigidity [9], which was proved by Hirzebruch [4], Krichever [5], etc. In this note, we prove the rigidity of the elliptic genus of level N on π_2-finite complex manifolds M for all $N > 0$ (see Theorem 2.1), which in turn implies the vanishing of the Hilbert polynomial $\chi(M, K^k) = 0$ for all k, where K denotes the canonical bundle of M (see Corollary 3.1).

The note is organized as follows: in Section 2 we give the definition of the elliptic genus of level N and state the Rigidity Theorem 2.1, and in Section 3 we sketch its proof.

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[1] Partially supported by a JSPS Research Fellowship PE-05030, PSC-CUNY award #67300-00-36, Convenio CONCYTEG 05-02-K117-112, and Apoyo CONACYT J48320-F.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2007.01.020

2. Rigidity of the elliptic genera of level N

Let M be a d-dimensional compact manifold and T its holomorphic tangent bundle. The elliptic genus of level N defined by Witten has the following q-development in the standard cusp of $\Gamma_1(N) \subset SL_2(\mathbb{Z})$, which we shall take as its definition

$$\tilde{\varphi}_N(M) = \sum_{j=0}^{\infty} \chi_y(M, R_j) q^j,$$

where $-y = \zeta = \mathrm{e}^{2\pi \mathrm{i}/N}$, and the R_j denote virtual vector bundles with coefficients in $\mathbb{Z}[\zeta]$ arising from the following infinite product

$$R(q,T) = \sum_{j=0}^{\infty} R_j q^j = \bigotimes_{j=1}^{\infty} \bigwedge\nolimits_{yq^j} T^* \otimes \bigotimes_{j=1}^{\infty} \bigwedge\nolimits_{y^{-1}q^j} T^* \otimes \bigotimes_{j=1}^{\infty} S_{q^j}(T + T^*),$$

where

$$\bigwedge\nolimits_t(W) = \sum_{j=0}^{\mathrm{rk}(W)} \bigwedge\nolimits^j W \cdot t^j \quad \text{and} \quad S_t(W) = \sum_{j=0}^{\infty} S^j W \cdot t^j$$

denote the sums of exterior and symmetric powers of a vector bundle W, respectively. The first two terms are

$$R_0 = 1, \qquad R_1 = (1-\zeta)T^* + \left(1 - \zeta^{-1}\right)T.$$

Thus we can see that this q-development has integral coefficients. The first term of $\tilde{\varphi}(M)$ is $\chi_y(M)$.

The q-developments at other cusps, however, have coefficients which are not necessarily integral. Such coefficients are of the form $\chi(M, K^{k/N} \otimes W_n)$ for some virtual vector bundle W_n, and the non-integrality may happen due to K not necessarily admitting an N-th root. For instance, the first term of the expansion at a cusp of the form $2\pi ik\tau/N$ for $1 \leqslant k \leqslant N$ is

$$\frac{1}{\tilde{q}^{k(N-k)d/2N}} \chi\left(M, K^{k/N}\right), \tag{1}$$

where $\tilde{q}$ is a uniformizing parameter for this cusp ($\tilde{q}^N = q$).

If we assume that M admits a holomorphic S^1-action, there is an induced action on the bundles R_j and on the cohomology groups $H^r(M, \bigwedge^p T^* \otimes R_j)$. Thus, the traces of such action on the cohomology groups produce the S^1-character $\chi(M, \bigwedge^p T^* \otimes R_j, \lambda)$, where $\lambda \in S^1$, so that

$$\chi_y(M, R_j, \lambda) = \sum_{p=0}^{d} \chi\left(M, \bigwedge\nolimits^p T^* \otimes R_j, \lambda\right) y^p, \qquad \tilde{\varphi}(M, \lambda) = \sum_{j=0}^{\infty} \chi_y(M, R_j, \lambda) q^j.$$

The rigidity of the elliptic genus for the S^1-action means that the finite Laurent series $\chi(M, R_j, \lambda)$ does not depend on λ and, therefore, $\tilde{\varphi}(M, \lambda)$ is constant in that variable. Thus, we can now state the rigidity theorem:

Theorem 2.1. *Let M be a compact complex manifold with finite second homotopy group, and admitting a non-trivial holomorphic S^1-action. Then, the equivariant elliptic genus $\tilde{\varphi}_N(M, \lambda)$ does not depend on $\lambda \in S^1$, i.e.*

$$\tilde{\varphi}_N(M, \lambda) = \tilde{\varphi}_N(M).$$

3. Sketch of proof

Hirzebruch's proof of the rigidity theorem [4] for the elliptic genus of level N considers a normalized version, applies the Atiyah–Bott–Singer fixed point theorem (holomorphic Lefschetz theorem) and examines the behaviour of the resulting meromorphic expressions. The normalized elliptic genus is

$$\varphi(M, \lambda) = \frac{\tilde{\varphi}(M, \lambda)}{\Upsilon(-\alpha)^d} = \sum_{j=0}^{\infty} \chi_y(M, S_j, \lambda) q^j,$$

where S_j are virtual vector bundles with coefficients in $\mathbb{Q}(\zeta)$,

$$\Upsilon(x) = \left(1 - \mathrm{e}^{-x}\right) \prod_{j=1}^{\infty} \frac{(1 - q^j \mathrm{e}^{-x})(1 - q^j \mathrm{e}^{x})}{1 - q^j},$$

$\alpha = 2\pi\mathrm{i}/N$. By applying the Atiyah–Singer–Bott fixed point theorem

$$\varphi(M, \lambda) = \sum_{\nu} \varphi_N(M, \lambda)_\nu,$$

where ν is an index for the connected components $M_\nu^{S^1}$ of M^{S^1},

$$\varphi_N(M, \lambda)_\nu = \left\langle e_0 \cdot F(x_1 + 2\pi\mathrm{i} m_1 z) \cdots F(x_d + 2\pi\mathrm{i} m_d z), \left[M_\nu^{S^1}\right]\right\rangle$$

e_0 is the Euler class of $M_\nu^{S^1}$, $F(x) = \Upsilon(x - \alpha)/(\Upsilon(x)\Upsilon(-\alpha))$, the m_i are the exponents of the infinitesimal action of S^1 on $T|_{M^{S^1}} = L^{m_1} \oplus \cdots \oplus L^{m_d}$, and x_i is the formal root of each one of the lines into which T splits.

If the first Chern class of M is divisible by N then

(i) $\varphi_N(M, \lambda)$ is elliptic with respect to a certain lattice;
(ii) $\varphi_N(M, \lambda)$ has no poles, which implies it is holomorphic and, therefore, constant in λ.

For (i), what is really needed is the S^1-action to be N-balanced. A circle action is called *N-balanced* if the residue class of the sum

$$m_1 + \cdots + m_d \pmod N$$

does not depend on the connected component $M_\nu^{S^1}$. The common residue is called the type t of the S^1-action.

Theorem 3.1. [4, p. 179] *For an N-balanced S^1-action of type t on the complex manifold M, the equivariant elliptic genus $\varphi_N(M \cdot \lambda)$, with $\lambda = \mathrm{e}^{2\pi\mathrm{i}z}$, is an elliptic function for the lattice $\mathbb{Z} \cdot N\tau + \mathbb{Z}$ which satisfies*

$$\varphi_N(M, \lambda q) = \zeta^t \varphi_N(M, \lambda), \quad (\zeta = \mathrm{e}^{2\pi\mathrm{i}/N}).$$

For (ii), we have to consider the sums

$$\psi(\lambda) = \sum_{M_\nu^{S^1} \subset X} \varphi_N(M, \lambda)_\nu$$

for those $M_\nu^{S^1}$ contained in a given connected component X of the fixed point set $M^{\mathbb{Z}_m}$, $\mathbb{Z}_m \subset S^1$, for every $m \in \mathbb{Z}$. Hirzebruch determined that $\psi(\lambda q^{s/m})$, for any integer s, has no poles on the unit circle as long as the residues

$$\sum_{i=1}^{d} \left[\frac{m_i}{m}\right] \pmod{mN}$$

are all equal. In this way, $\varphi_N(M, \lambda)$ has no poles at all and the rigidity theorem follows if $c_1(M) \equiv 0 \pmod N$

$$\varphi_N(M, \lambda) = \varphi_N(M).$$

However, conditions (i) and (ii) on the S^1-action *are also fulfilled* by actions on complex manifolds with finite second homotopy group. Consider the S^1-decompositions of the tangent space at two distinct S^1-fixed points p and p' in terms of generator $L \cong \mathbb{C}$ of the representation ring $R(S^1)$

$$T_p M = L^{m_1} \oplus \cdots \oplus L^{m_d}, \qquad T_{p'} M = L^{m'_1} \oplus \cdots \oplus L^{m'_d},$$

where m_i and m'_i are the exponents of the S^1 action at p and p', respectively. By [2], the virtual representation $T_p - T_{p'}$ can be factored as follows

$$T_p - T_{p'} = (1 - L)^2 \otimes \left(\bigoplus_j b_j L^j\right),$$

where the set $\{b_j \in \mathbb{Z}\}$ is finite. Thus

$$\sum_{i=1}^{d} m_i - \sum_{i=1}^{d} m_i' = \sum_j b_j \cdot j - 2\sum_j b_j \cdot (j+1) + \sum_j b_j \cdot (j+2) = 0.$$

Hence, conditions (i) and (ii) hold, and the Rigidity Theorem follows.

As a consequence, we see that

$$\varphi_N(M) = \varphi_N(M, \lambda q) = \zeta^t \varphi_N(M, \lambda) = \zeta^t \varphi_N(M),$$

so that, if the S^1-action has type $t \neq 0$ then

$$\varphi_N(M) \equiv 0 \quad \text{and} \quad \tilde{\varphi}_N(M) \equiv 0.$$

On the other hand, the Rigidity Theorem readily implies that the not necessarily integral characteristic numbers

$$\chi(M, K^{k/N}) = 0$$

for $k = 1, \ldots, N-1$, as in [4]. Since we have imposed *no divisibility condition* on the first Chern class of M, these vanishings hold for any N. Thus, the Hilbert polynomial $\chi(M, K^k)$ has infinitely many zeroes and is, therefore, identically zero.

Corollary 3.1. *Let M be a compact complex manifold with finite second homotopy group, and admitting a non-trivial holomorphic S^1-action. Then*

$$\chi(M, K^k) = 0 \quad \textit{for all } k.$$

In particular, the Todd genus vanishes, Todd$(M) = 0$. □

Hence, the Todd genus is an obstruction to the existence of holomorphic circle actions on π_2-finite compact complex manifolds.

Acknowledgement

The author wishes to thank the Max Planck Institute for Mathematics (Bonn) for its hospitality and support.

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