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Algebraic Geometry

Local structure of $\mathcal{SU}_C(3)$ for a curve of genus 2

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Abstract

The aim of this Note is to give a precise description of the local structure of the moduli space $\mathcal{SU}_C(3)$ of rank 3 vector bundles on a curve C of genus 2, which is in particular shown to be a local complete intersection. This allows us to investigate the local structure of the branch locus of the theta map, the dual of which is known to be the Coble cubic in $\mathbb{P}H^0(J_C^1, 3\Theta)$. ***To cite this article: O. Serman, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Structure locale de $\mathcal{SU}_C(3)$ pour une courbe de genre 2. Le but de cette Note est de donner une description précise de la structure locale en tout point de l'espace de modules $\mathcal{SU}_C(3)$ des fibrés vectoriels de rang 3 sur une courbe de genre 2. Cette étude montre notamment que cet espace est localement intersection complète. Elle permet aussi d'analyser la structure locale du lieu de branchement de l'application thêta, qui n'est autre que la variété duale de la cubique de Coble dans $\mathbb{P}H^0(J_C^1, 3\Theta)$. ***Pour citer cet article : O. Serman, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Soit C une courbe projective lisse de genre 2 définie sur un corps algébriquement clos k de caractéristique nulle. Notons $\mathcal{SU}_C(3)$ l'espace de modules des fibrés vectoriels semi-stables de rang 3 sur C dont le déterminant est trivial. Laszlo a considéré dans [4] la structure locale de cet espace ; le point de départ de cette étude est le théorème des slices étales de Luna qui assure que, au voisinage d'un point défini par un fibré E, l'espace $\mathcal{SU}_C(3)$ est localement isomorphe (pour la topologie étale) au quotient $\mathrm{Ext}^1(E, E)_0 /\!/ \mathrm{Aut}(E)$ à l'origine, où $\mathrm{Ext}^1(E, E)_0$ désigne le noyau de $\mathrm{tr}: \mathrm{Ext}^1(E, E) \to H^1(C, \mathcal{O}_C)$.

La décomposition (2) montre comment interpréter ce quotient en termes de représentations d'un carquois Q. Cette traduction permet de donner une description explicite de $\mathrm{Ext}^1(E, E)_0 /\!/ \mathrm{Aut}(E)$ pour tout point $E \in \mathcal{SU}_C(3)$.

Considérons par exemple le cas d'un fibré E de la forme $E = (L \otimes V) \oplus L^{-2}$ avec L fibré inversible de degré 0 tel que $L^3 \not\simeq \mathcal{O}$ et V espace vectoriel de dimension 2. Il s'agit d'après ce qui précède de calculer l'algèbre des polynômes invariants sur l'espace des représentations de dimension $(2, 1)$ du carquois (4). Ce point

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de vue conduit à la discussion suivante : le $\mathrm{Aut}(E)$-module $\mathrm{Ext}^1(E,E)_0$ est exactement le $\mathbf{GL}(V)\times\mathbb{G}_m$-module $(H^1(C,\mathcal{O})\otimes\mathrm{End}(V))\oplus(H^1(C,L^{-3})\otimes V^*)\oplus(H^1(C,L^3)\otimes V)$, de sorte qu'en choisissant des bases pour chacun des espaces de cohomologie considérés on peut écrire tout élément de $\mathrm{Ext}^1(E,E)_0$ sous la forme $(a_1,a_2,\lambda,v)\in\mathrm{End}(V)\oplus\mathrm{End}(V)\oplus V^*\oplus V$. L'application $(a_1,a_2,\lambda,v)\mapsto(a_1,a_2,a_3=\lambda\otimes v)\in\mathrm{End}(V)^{\oplus3}$ identifie alors le quotient $\mathrm{Ext}^1(E,E)_0/\!/\mathrm{Aut}(E)$ au fermé de $\mathrm{End}(V)^{\oplus3}/\!/\mathbf{GL}(V)$ défini par l'équation $\det(a_3)=0$.

Une présentation de l'anneau de fonctions de ce dernier quotient est donnée dans [2]. Cette présentation permet de conclure l'étude locale au voisinage du point E considéré ici : l'espace $\mathcal{SU}_C(3)$ est localement isomorphe en E (pour la topologie étale) à un voisinage de l'origine dans le fermé de $\mathbb{A}^{10}$ défini par les deux équations

$$X_{10}^2+18\bigl(X_4X_5X_6+2X_7X_8X_9-X_6X_7^2-X_5X_8^2-X_4X_9^2\bigr)=0\quad\text{et}\quad X_3^2-2X_6=0.$$

On en déduit notamment que le cône tangent en un tel point est un hyperplan double dans $\mathbb{A}^9$, ce qui précise le résultat de [4, V].

L'étude détaillée des autres cas montre en particulier le résultat suivant :

Théorème 0.1. *L'espace $\mathcal{SU}_C(3)$ est localement intersection complète.*

Soit Θ le diviseur canonique sur la variété J^1 paramétrant les fibrés en droites sur C de degré 1 ; l'application thêta $\theta:\mathcal{SU}_C(3)\to|3\Theta|$ est un morphisme de degré 2. Ortega a montré dans [6] que le lieu de branchement $\mathcal{S}\subset|3\Theta|$ de θ est l'hypersurface sextique duale de la cubique de Coble $\mathcal{C}\subset|3\Theta|^*$.

D'après [6] l'involution σ associée au morphisme θ est $E\mapsto\iota^*E^*$ où ι désigne l'involution hyperelliptique de C. Il est alors aisé de déduire de l'étude précédente la structure locale de la sextique $\mathcal{S}$: il suffit en effet d'expliciter l'action de la linéarisation de σ via les morphismes étales donnés par le théorème de Luna.

À titre d'exemple considérons à nouveau le cas d'un fibré $E=(L\otimes V)\oplus L^{-2}$ avec $L^3\not\simeq\mathcal{O}$ (notons qu'un tel fibré appartient toujours à $\mathcal{S}$). Si l'on choisit les éléments non nuls de $H^1(C,L^3)$ et $H^1(C,L^{-3})$ introduits plus haut de manière à ce qu'ils correspondent via $\sigma:\mathrm{Ext}^1(L^{-2},L)\xrightarrow{\sim}\mathrm{Ext}^1(L,L^{-2})$, l'opération de σ sur $\mathrm{Ext}^1(E,E)_0$ s'écrit de la façon suivante

$$(x,y,\lambda,v)\in\mathrm{End}(V)^{\oplus2}\oplus V^*\oplus V\mapsto({}^tx,{}^ty,{}^tv,{}^t\lambda)\ ;$$

on en déduit immédiatement l'action de σ sur les générateurs de $k[\mathrm{Ext}^1(E,E)_0/\!/\mathrm{Aut}(E)]$ donnés en (5), puis la structure locale de $\mathcal{S}$ en E : la sextique est localement isomorphe en ce point au fermé de $\mathbb{A}^9$ défini par les équations $X_4X_5X_6+2X_7X_8X_9-X_6X_7^2-X_5X_8^2-X_4X_9^2=0$ et $X_3^2-2X_6=0$. Son cône tangent est l'hypersurface cubique de $\mathbb{A}^8$ définie par $2X_7X_8X_9-X_5X_8^2-X_4X_9^2=0$.

1. Introduction

Let C be a smooth irreducible projective curve of genus 2 over an algebraically closed field k of characteristic zero, and let $\mathcal{SU}_C(3)$ be the moduli space of rank 3 vector bundles on C with trivial determinant. Laszlo began to investigate the local structure of this moduli space in [4, V]: Luna's étale slice theorem provides a way to compute the completed local ring at any point of $\mathcal{SU}_C(3)$ as GIT quotients of affine spaces, but, as soon as the isotropy group gets too bad, this leads to a quite intricate calculation. By translating this situation in terms of representations of quivers, we managed to work out the local structure at any point of $\mathcal{SU}_C(3)$. We have, in particular, obtained the following result:

Theorem 1.1. *The moduli space of rank* 3 *vector bundles on a curve of genus* 2 *is a local complete intersection.*

As we have already seen in [7], the notion of representations of quivers appears to be really helpful to understand the quotients given by Luna's result. Although it may not be clear in this Note, where we could have given direct proofs avoiding such considerations, this quiver setting was the very basic point which led to generating sets for the coordinate rings of the quotients.

Let now Θ be the canonical Theta divisor on the variety J^1 which parametrizes line bundles of degree 1 on C. It is known for long that the theta map $\theta:\mathcal{SU}_C(3)\to|3\Theta|$ is a double covering. Ortega has shown in [6] that its branch locus $\mathcal{S}\subset|3\Theta|$ is a sextic hypersurface which is the dual of the Coble cubic $\mathcal{C}\subset|3\Theta|^*$ (recall that the Coble cubic is the unique cubic in $|3\Theta|^*$ which is singular along $J^1 \xrightarrow{|3\Theta|} |3\Theta|^*$). The last part of this Note is devoted to the local structure of the sextic $\mathcal{S}$.

2. Local structure of $\mathcal{SU}_C(3)$

The starting point of the local study of moduli spaces of vector bundles, which follows from Luna's slice theorem, can be found in [4, II]: it states that, at a closed point representing a polystable bundle E, the moduli space $\mathcal{SU}_C(r)$ of rank r vector bundles with trivial determinant is étale locally isomorphic to the quotient $\mathrm{Ext}^1(E,E)_0/\!\!/\mathrm{Aut}(E)$ at the origin, where $\mathrm{Ext}^1(E,E)_0$ is the kernel of $\mathrm{tr}:\mathrm{Ext}^1(E,E)\to H^1(C,\mathcal{O}_C)$.

We thus have to understand the ring of invariants of $k[\mathrm{Ext}^1(E,E)_0]=\mathrm{Sym}(\mathrm{Ext}^1(E,E)_0^*)$ under the action of $\mathrm{Aut}(E)$. As a polystable bundle E can be written

$$E=\bigoplus_{i=1}^{s} E_i\otimes V_i, \tag{1}$$

where the E_i's are mutually non-isomorphic stable bundles (of rank r_i and degree 0), and the V_i's are vector spaces (of dimension ρ_i). Through this splitting our data become

$$\mathrm{Ext}^1(E,E)=\bigoplus_{i,j}\mathrm{Ext}^1(E_i,E_j)\otimes \mathrm{Hom}(V_i,V_j), \tag{2}$$

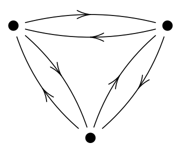
endowed with an operation of $\mathrm{Aut}(E)=\prod_i \mathbf{GL}(V_i)$ coming from the natural actions of $\mathbf{GL}(V_i)\times\mathbf{GL}(V_j)$ on $\mathrm{Hom}(V_i,V_j)$.

We recognize here the setting of representations of quivers (we refer to [5] for the definitions and notations): consider indeed the quiver Q with s vertices $1,\dots,s$, and $\dim\mathrm{Ext}^1(E_i,E_j)$ arrows from i to j, and define $\alpha\in\mathbb{N}^s$ by $\alpha_i=\rho_i$. The $\mathrm{Aut}(E)$-module $\mathrm{Ext}^1(E,E)$ is then exactly the $\mathbf{GL}(\alpha)$-module $R(Q,\alpha)$ consisting of all representations of Q of dimension α. This point of view identifies the quotient $\mathrm{Ext}^1(E,E)_0/\!\!/\mathrm{Aut}(E)$ we have in mind with a closed subscheme of $R(Q,\alpha)/\!\!/\mathbf{GL}(\alpha)$, and [5] shows that the coordinate ring of the latter is generated by traces along oriented cycles in the quiver Q. But we also need a precise description of the relations between these generators (the *second main theorem for invariant theory*). Once we have a convenient enough statement about these relations we can describe the completed local ring of $\mathcal{SU}_C(r)$ at E.

When $r=3$ the decomposition (1) ensures that there are only five cases to deal with, according to the values of the r_i's and ρ_i's.

2.1. The case of a stable bundle is obvious, and the case $r_1=2, r_2=1$ is a special case of the situation studied in [4, III]: $\mathcal{SU}_C(3)$ is étale isomorphic at E to a rank 4 quadric in $\mathbb{A}^9$. Here quivers do not provide a shorter proof.

2.2. Let us look at the three other cases, where every E_i in (1) is invertible. The generic case consists of bundles E which are direct sum of 3 distinct line bundles. It has already been performed in [4, V], but may also be recovered in a more convenient fashion as an easy consequence of [5]: the generators of [4, Lemma V.1] then arise nicely as traces along closed cycles in the quiver

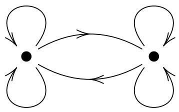


(3)

(note that there should be two loops on each vertex; but $\alpha=(1,1,1)$ implies that we can restrict ourselves to the quiver (3)). It is easy too to infer from (3) the relation found by Laszlo; but, although [5] gives a way to produce all the relations, this description turns out to be quite inefficient even in the present case (note however that, in order to conclude here, it is enough to remind that we know a priori the dimension of $\mathrm{Ext}^1(E,E)/\!\!/\mathrm{Aut}(E)$).

In the remaining two cases we already know that the tangent cone at E must be a quadric (in $\mathbb{A}^9$) of rank $\leqslant 2$ (see [4, V]). We give now more precise statements.

2.3. Suppose that $\rho_1 = 2$, i.e. that $E = (L \otimes V) \oplus L^{-2}$ where L is a line bundle of degree 0 with $L^3 \not\simeq \mathcal{O}$ and V a vector space of dimension 2. We have to consider here the ring of invariant polynomials on the representation space $R(Q, (2, 1))$ of the quiver Q



(4)

under the action of $\mathbf{GL}(V) \times \mathbb{G}_m$. Since the second vertex corresponds to a 1-dimensional vector space it is enough to consider the quiver obtained by deleting the two loops on the right, and in fact we are brought to the action of $\mathbf{GL}(V)$ on $\mathrm{End}(V) \oplus \mathrm{End}(V) \oplus \mathrm{End}(V)^{\leqslant 1} \subset \mathrm{End}(V)^{\oplus 3}$, where $\mathrm{End}(V)^{\leqslant 1}$ denotes the space of endomorphisms of V of rank at most 1: this simply means that

$$k\big[R\big(Q, (2,1)\big)\big]^{\mathbf{GL}(V)\times\mathbb{G}_m} \simeq \big(k\big[R\big(Q,(2,1)\big)\big]^{\mathbb{G}_m}\big)^{\mathbf{GL}(V)},$$

and that $k[R(Q, (2, 1))]^{\mathbb{G}_m}$ gets naturally identified (as a $\mathbf{GL}(V)$-module) with

$$\mathrm{End}(V) \oplus \mathrm{End}(V) \oplus \mathrm{End}(V)^{\leqslant 1} \oplus k \oplus k,$$

the last two summands being fixed under the induced operation of $\mathbf{GL}(V)$.

We make now this discussion more concrete in coming back to $\mathrm{Ext}^1(E, E)_0 /\!\!/ \mathrm{Aut}(E)$. Since (2) here reads

$$\mathrm{Ext}^1(E,E) = \big(H^1(C,\mathcal{O}) \otimes \mathrm{End}(V)\big) \oplus \big(H^1\big(C, L^{-3}\big) \otimes V^*\big) \oplus \big(H^1\big(C, L^3\big) \otimes V\big) \oplus \big(H^1(C,\mathcal{O}) \otimes k\big),$$

we can identify the $\mathrm{Aut}(E)$-module $\mathrm{Ext}^1(E, E)_0$ with the $\mathbf{GL}(V) \times \mathbb{G}_m$-module

$$\big(H^1(C,\mathcal{O}) \otimes \mathrm{End}(V)\big) \oplus \big(H^1\big(C, L^{-3}\big) \otimes V^*\big) \oplus \big(H^1\big(C, L^3\big) \otimes V\big),$$

so that, up to the choices of some basis of the different cohomology spaces, any element of $\mathrm{Ext}^1(E, E)_0$ can be written $(a_1, a_2, \lambda, v) \in \mathrm{End}(V) \oplus \mathrm{End}(V) \oplus V^* \oplus V$. The map $(a_1, a_2, \lambda, v) \mapsto (a_1, a_2, a_3 = \lambda \otimes v) \in \mathrm{End}(V)^{\oplus 3}$ identifies the quotient $\mathrm{Ext}^1(E, E)_0 /\!\!/ \mathrm{Aut}(E)$ with the closed subscheme of $\mathrm{End}(V)^{\oplus 3} /\!\!/ \mathbf{GL}(V)$ defined by the equation $\det a_3 = 0$. A presentation of $k[\mathrm{End}(V)^{\oplus 3}]^{\mathbf{GL}(V)}$ can be found in [2] (note that another presentation of this ring had been previously given in [3]): if we let b_i denote the traceless endomorphism $a_i - \frac{1}{2}\mathrm{tr}(a_i)\mathrm{id}$, this invariant ring is generated by the following ten functions

$$u_i = \mathrm{tr}(a_i) \quad \text{with } 1 \leqslant i \leqslant 3, \quad v_{ij} = \mathrm{tr}(b_i b_j) \quad \text{with } 1 \leqslant i \leqslant j \leqslant 3, \quad w = \sum_{\sigma \in \mathfrak{S}_3} \varepsilon(\sigma)\,\mathrm{tr}(b_{\sigma(1)} b_{\sigma(2)} b_{\sigma(3)}), \tag{5}$$

subject to the single relation $w^2 + 18\det(v_{ij}) = 0$. We have thus obtained the following result:

Proposition 2.4. *If $E = (L \otimes V) \oplus L^{-2}$ with $L^3 \not\simeq \mathcal{O}$, then $\mathcal{SU}_C(3)$ is étale locally isomorphic at E with the subscheme of $\mathbb{A}^{10}$ defined by the two equations*

$$X_{10}^2 + 18\big(X_4X_5X_6 + 2X_7X_8X_9 - X_6X_7^2 - X_5X_8^2 - X_4X_9^2\big) = 0 \quad \textit{and} \quad X_3^2 - 2X_6 = 0$$

at the origin. Its tangent cone is a double hyperplane in $\mathbb{A}^9$.

2.5. Suppose now that $\rho_1 = 3$, i.e. that $E = L \otimes V$ where V is a vector space of dimension 3 (and L a line bundle of order 3). By the same argument as in [4, Proposition V.4] we know that the tangent cone at such a point is a rank 1 quadric. But an explicit description of an étale neighbourhood is available, thanks to [1]. The space $\mathrm{Ext}^1(E, E)_0$ is isomorphic to $H^1(C, \mathcal{O}) \otimes \mathrm{End}_0(V)$ and, if we fix a basis of $H^1(C, \mathcal{O})$, any of its element can be written $(x, y) \in \mathrm{End}_0(V) \oplus \mathrm{End}_0(V)$. The ring of invariants $k[H^1(C, \mathcal{O}) \otimes \mathrm{End}_0(V)]^{\mathbf{GL}(V)}$ is then generated by the nine functions $\mathrm{tr}(x^2)$, $\mathrm{tr}(xy)$, $\mathrm{tr}(y^2)$, $\mathrm{tr}(x^3)$, $\mathrm{tr}(x^2y)$, $\mathrm{tr}(xy^2)$, $\mathrm{tr}(y^3)$, $v = \mathrm{tr}(x^2y^2) - \mathrm{tr}(xyxy)$ and $w = \mathrm{tr}(x^2y^2xy) - \mathrm{tr}(y^2x^2yx)$; moreover the ideal of relations is principal, generated by an explicit equation [1].

As a result of this case-by-case analysis we conclude that $\mathcal{SU}_C(3)$ is a local complete intersection, as announced in the introduction.

3. On the local structure of $\mathcal{S}$

We know from [6] that the involution σ associated to the double covering $\theta : \mathcal{SU}_C(3) \to |3\Theta|$ acts by $E \mapsto \iota^* E^*$, where ι stands for the hyperelliptic involution. The local study of its ramification locus thus reduces to an explicit analysis of the behaviour of σ through the étale morphisms resulting from Luna's theorem. Once again it comes to a case-by-case investigation.

3.1. When E is stable there is nothing to say. If $E = F \oplus L$ (with F a stable bundle of rank 2 and $L = (\det F)^{-1}$) we have to understand the action of the linearization of σ on

$$\mathrm{Ext}^1(E,E)_0 \simeq \mathrm{Ext}^1(F,F) \oplus \mathrm{Ext}^1(F,L) \oplus \mathrm{Ext}^1(L,F)$$

(we have identified $\mathrm{Ext}^1(F,F)$ with its image in

$$\mathrm{Ext}^1(F,F) \oplus H^1(C,\mathcal{O}) \subset \mathrm{Ext}^1(E,E)$$

by the map $\omega \mapsto (\omega, -\mathrm{tr}(\omega))$).

Since $\sigma(E) = E$, $\iota^* F^*$ must be isomorphic to F, and σ identifies $\mathrm{Ext}^1(F,L)$ and $\mathrm{Ext}^1(L,F)$; let us choose a basis X_1, X_2 of $\mathrm{Ext}^1(F,L)^*$, and call Y_1, Y_2 the corresponding basis of $\mathrm{Ext}^1(L,F)$. We need here to recall precisely from [4] the explicit description of the coordinate ring of $\mathrm{Ext}^1(E,E)_0/\!\!/\mathrm{Aut}(E)$ mentioned in 2.1: it is generated by $k[\mathrm{Ext}^1(F,F)]$ and the four functions $u_{ij} = X_i Y_j$, subject to the relation $u_{11}u_{22} - u_{12}u_{21} = 0$.

It follows from our choice that σ maps u_{ij} to u_{ji}. Furthermore we claim that σ acts identically on $\mathrm{Ext}^1(F,F)$: as a stable bundle, F corresponds to a point of the moduli space $\mathcal{U}(2,0)$, whose tangent space is precisely isomorphic to $\mathrm{Ext}^1(F,F)$. The action of σ on this vector space is the linearization of the one of $F \in \mathcal{U}(2,0) \mapsto \iota^* F^*$. Using that $\mathcal{U}(2,0)$ is a Galois quotient of $J_C \times \mathcal{SU}_C(2)$, our claim comes from the fact that σ is trivial on both J_C and $\mathcal{SU}_C(2)$.

Since the coordinate ring of the fixed locus of σ in $\mathrm{Ext}^1(E,E)_0/\!\!/\mathrm{Aut}(E)$ is the quotient of the one of $\mathrm{Ext}^1(E,E)_0/\!\!/\mathrm{Aut}(E)$ by the involution induced by σ we may conclude that $\mathcal{S}$ is étale locally isomorphic at E to the quadric cone in $\mathbb{A}^8$ defined by $X_3^2 - X_1X_2 = 0$.

3.2. Consider now the situation of Section 2.2: let us write $E = L_1 \oplus L_2 \oplus L_3$ with $L_i \not\simeq L_j$ if $i \neq j$. We have $\mathrm{Ext}^1(E,E) \simeq \bigoplus_{i,j} \mathrm{Ext}^1(L_i,L_j)$; let us choose for $i \neq j$ a non-zero element X_{ij} of $\mathrm{Ext}^1(L_i,L_j)^*$ such that X_{ji} corresponds to X_{ij} through the isomorphism $\mathrm{Ext}^1(L_i,L_j) \simeq \mathrm{Ext}^1(L_j,L_i)$ induced by σ and the natural isomorphisms $\iota^* L_i^* \simeq L_i$. It then follows from Section 2.2 (see [4] for a complete proof) that the ring $k[\mathrm{Ext}^1(E,E)_0]^{\mathrm{Aut}(E)}$ is generated by $k[\ker(\bigoplus_i \mathrm{Ext}^1(L_i,L_i) \to H^1(C,\mathcal{O}))]$ and the five functions $Y_1 = X_{23}X_{32}$, $Y_2 = X_{13}X_{31}$, $Y_3 = X_{12}X_{21}$, $Y_4 = X_{12}X_{23}X_{31}$, $Y_5 = X_{13}X_{32}X_{21}$, subject to the relation $Y_4Y_5 - Y_1Y_2Y_3 = 0$. The involution σ fixes $k[\ker(\bigoplus_i \mathrm{Ext}^1(L_i,L_i) \to H^1(C,\mathcal{O}))]$, Y_1, Y_2 and Y_3, while it sends Y_4 to Y_5. The fixed locus $\mathrm{Fix}(\sigma)$ is then defined by the equation $Y_4 - Y_5 = 0$, so that $\mathcal{S}$ is étale locally isomorphic to the hypersurface in $\mathbb{A}^8$ defined by $Z_4^2 - Z_1Z_2Z_3 = 0$. Its tangent cone is a double hyperplane.

3.3. In the situation of 2.3 we have to make a more precise choice of the non-zero elements of $\mathrm{Ext}^1(L^{-2},L)$ and $\mathrm{Ext}^1(L,L^{-2})$, so as to make them correspond through σ and the natural isomorphism $\iota^* L^* \simeq L$; such a choice ensures that σ operates on $\mathrm{Ext}^1(E,E)_0$ in the following way:

$$(x,y,\lambda,v) \in \mathrm{End}(V)^{\oplus 2} \oplus V^* \oplus V \mapsto \left({}^t x, {}^t y, {}^t v, {}^t \lambda\right),$$

so that we know how it acts on the generators of $k[\mathrm{Ext}^1(E,E)_0/\!\!/\mathrm{Aut}(E)]$ given in (5): σ fixes u_i, v_{ij}, and sends w to $-w$. This implies that the fixed locus is defined by the equation $w = 0$. The sextic $\mathcal{S}$ is étale locally isomorphic to the subscheme of $\mathbb{A}^9$ whose ideal is generated by the two equations

$$X_4X_5X_6 + 2X_7X_8X_9 - X_6X_7^2 - X_5X_8^2 - X_4X_9^2 = 0 \quad \text{and} \quad X_3^2 - 2X_6 = 0;$$

its tangent cone is therefore the cubic hypersurface of $\mathbb{A}^8$ defined by $2X_7X_8X_9 - X_5X_8^2 - X_4X_9^2 = 0$.

3.4. We are now left with the last case, where E is of the form $L \otimes V$ (with $L^3 = \mathcal{O}$): $\mathrm{Ext}^1(E,E)_0$ is then isomorphic to $H^1(C,\mathcal{O}) \otimes \mathrm{End}_0(V)$, and σ acts by $\omega \otimes a \in H^1(C,\mathcal{O}) \otimes \mathrm{End}_0(V) \mapsto \omega \otimes {}^t a$. This induces an action on

$k[H^1(C,\mathcal{O})\otimes \mathrm{End}_0(V)]^{\mathrm{Aut}(E)}$ which fixes the first eight generators of 2.5, and acts by -1 on the last one, namely w; the fixed locus is thus defined in $\mathrm{Ext}^1(E,E)_0/\!\!/\,\mathrm{Aut}(E)$ by the linear equation $w=0$.

The sextic $\mathcal{S}$ is then étale locally isomorphic to an hypersurface in $\mathbb{A}^8$ defined by an explicit equation; writing down this equation shows that its tangent cone is a triple hyperplane.

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