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Mathematical Problems in Mechanics

Influence of interfacial pressure on the hyperbolicity of the two-fluid model

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Abstract

The two-fluid model, widely used in the modeling of two phase flows, generally fails to be hyperbolic in its basic formulation. However, it is well known that interfacial forces, bringing new differential terms to the system can make the model hyperbolic. This Note details the effects interfacial pressure has on the hyperbolicity of the two-fluid model, in the general case of two compressible phases. We characterise the location and topology of the non-hyperbolic regions and propose a closure law for the interfacial pressure to make the system hyperbolic. ***To cite this article: M. Ndjinga, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Influence de la pression interfaciale sur l'hyperbolicité du modèle bifluide. Les équations de base du modèle bifluide, couramment utilisées dans la modélisation des écoulements diphasiques ne sont en général pas hyperboliques. Cependant, les forces interfaciales apportant de nouveaux termes différentiels peuvent modifier ce résultat. Cette Note se propose de détailler l'effet du défaut de pression interfaciale sur l'hyperbolicité du modèle, dans le cas général de deux phases compressibles. Nous mettons en évidence la localisation et la topologie des régions non hyperboliques, et nous proposons une relation de fermeture de la pression interfaciale rendant le système hyperbolique. ***Pour citer cet article : M. Ndjinga, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Les équations de base du modèle bifluide (voir par exemple [2]) couramment utilisées dans la modélisation des écoulements compressibles à plusieurs phases, ne sont pas hyperboliques en général [6]. Les éventuelles solutions sont donc linéairement instables, et la simulation numérique fait alors apparaître des oscillations non physiques, indépendantes de la méthode numérique utilisée. La modélisation des forces d'interface peut cependant faire apparaître de nouveaux termes différentiels susceptibles de rendre le système hyperbolique. Il est courant d'utiliser un terme différentiel lié au défaut de pression interfaciale Δp afin de rendre le système hyperbolique (voir par exemple [1,3]).

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L'éfficacité de cette méthode n'a été montrée que dans le cas où les deux fluides sont supposés incompressibles (voir [7]). On analyse ici le caractère hyperbolique du modèle bifluide à 4 équations d'écoulement isentropique à deux phases compressibles, en présence du paramètre Δp.

L'analyse de l'hyperbolicité du système revient à décrire la topologie de la région où le polynôme caractéristique P_A du système (1)–(4) est scindé sur $\mathbb{R}$. Pour cela, nous montrons que le nombre de racines réelles de P_A est le nombre de points d'intersection de deux coniques (8) et (9). Nous caractérisons, pour K_1 et K_2 supposés constants, la topologie de la région où l'hyperbole et la parabole admettent 0, 2 ou 4 points d'intersection dans un repère (K, u_r^2) (Fig. 2). La courbe des racines doubles, qui délimite les régions hyperboliques et non hyperboliques, rencontre les axes principaux en quatre points dont les coordonnées sont calculées (Éqs. (10) et (11)). De cette étude nous déduisons le « diagramme d'hyperbolicité » du modèle bifluide dans un repère $(\Delta p, u_r^2)$ (Fig. 3). Une loi de fermeture de la pression interfaciale peut être considérée comme un couplage entre Δp et u_r^2, représentée par une courbe sur le diagrame d'hyperbolicité pouvant traverser une région non hyperbolique. Le diagramme d'hyperbolicité fait apparaître une zone d'étranglement de la région hyperbolique (segment $[A_1, B_1]$) entre deux régions non hyperboliques. Grâce au calcul des points clés du diagramme d'hyperbolicité (Éq. (12)), nous proposons une fermeture simple de pression interfaciale permettant de traverser cette zone d'étranglement et de se maintenir dans la région hyperbolique.

Le modèle général non isentropique à 6 équations, comprenant pour chaque phase une équation d'évolution de l'énergie, apporte de nouvelles valeurs propres triviales u_g et u_l au système (voir [6]). Les valeurs propres non triviales sont racines d'un polynôme ayant la même forme que le polynôme P_A (Éq. (5)) étudié dans cette Note, et l'étude présentée ici reste donc valable.

1. The characteristic polynomial

We consider a one-dimensional two-fluid model for a compressible isentropic two-phase flow (see [2]):

$$\frac{\partial \alpha_1\rho_1}{\partial t} + \frac{\partial \alpha_1\rho_1 u_1}{\partial x} = 0, \tag{1}$$

$$\frac{\partial \alpha_2\rho_2}{\partial t} + \frac{\partial \alpha_2\rho_2 u_2}{\partial x} = 0, \tag{2}$$

$$\frac{\partial \alpha_1\rho_1 u_1}{\partial t} + \frac{\partial \alpha_1\rho_1 u_1^2}{\partial x} + \alpha_1\frac{\partial p}{\partial x} + \Delta p\frac{\partial \alpha_1}{\partial x} = 0, \tag{3}$$

$$\frac{\partial \alpha_2\rho_2 u_2}{\partial t} + \frac{\partial \alpha_2\rho_2 u_2^2}{\partial x} + \alpha_2\frac{\partial p}{\partial x} + \Delta p\frac{\partial \alpha_2}{\partial x} = 0; \tag{4}$$

u_1 and u_2 denote the phasic velocities, and α_1 and α_2 are the phasic volume fractions, with the closure relation $\alpha_1 + \alpha_2 = 1$. The phasic densities ρ_1 and ρ_2 are strictly increasing differentiable functions of the bulk averaged pressure p. We define the phasic sound speeds $c_1 = (\partial\rho_1/\partial p)^{-1/2}$ and $c_2 = (\partial\rho_2/\partial p)^{-1/2}$, and assume $\rho_1 c_1^2 \leqslant \rho_2 c_2^2$; Δp is a quantity that takes into account the pressure default $p - p_i$ between the bulk average pressure, and the interfacial average pressure. Mass and momentum transfer terms between the phases have been neglected.

Defining the unknown vector $U = (\alpha_1\rho_1, \alpha_1\rho_1 u_1, \alpha_2\rho_2, \alpha_2\rho_2 u_2)$, the system (1)–(4) can be written in the matrix form

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = 0,$$

and is said to be strictly hyperbolic if $A(U)$ admits real distinct eigenvalues. Using the abbreviation

$$\gamma^2 = \frac{c_1^2 c_2^2}{\alpha_1\rho_2 c_2^2 + \alpha_2\rho_1 c_1^2},$$

we have

$$A(U) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha_1\rho_2\gamma^2 - u_1^2 & 2u_1 & \alpha_1\rho_1\gamma^2 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha_2\rho_2\gamma^2 & 0 & \alpha_2\rho_1\gamma^2 - u_2^2 & 2u_2 \end{pmatrix}.$$

Define the polynomial:

$$P_A(X)=\left(X-\frac{u_r}{2}\right)^2\left(X+\frac{u_r}{2}\right)^2-K_1\left(X-\frac{u_r}{2}\right)^2-K_2\left(X+\frac{u_r}{2}\right)^2+K_3, \tag{5}$$

with

$$K_1=\gamma^2\left(\alpha_2\rho_1+\frac{\alpha_1}{c_1^2}\Delta p\right),\quad K_2=\gamma^2\left(\alpha_1\rho_2+\frac{\alpha_2}{c_2^2}\Delta p\right),\quad K_3=\gamma^2\,\Delta p,\quad u_r=u_1-u_2. \tag{6}$$

One can check that $P_A(X+\frac{u_1+u_2}{2})$ is the characteristic polynomial of A. Therefore, the system (1)–(4) is strictly hyperbolic if and only if P_A admits 4 distinct real roots.

2. A geometric interpretation

A real root X of P_A can be associated to a unique couple

$$x=\left(X+\frac{u_r}{2}\right)^2,\qquad y=\left(X-\frac{u_r}{2}\right)^2, \tag{7}$$

and $P_A(X)=0$ can be rewritten:

$$(x-K_1)(y-K_2)=K_1K_2-K_3. \tag{8}$$

Moreover, x and y also satisfy:

$$2\,u_r^2(x+y)=(x-y)^2+u_r^4,\quad x\geqslant 0,\ y\geqslant 0. \tag{9}$$

Conversely if x and y satisfy (9) and (8), then there exists an X satisfying both equations in (7) and, from (8) we have that X is a root of P_A. Therefore the number of real roots of P_A is the number of intersecting points between the hyperbola and the parabola defined by Eqs. (8) and (9).

If the parameter $u_r^2\neq 0$, the parabola is located on the $x\geqslant 0$ and $y\geqslant 0$ quarter, and is tangent to both x and y axis at distance u_r^2. If $u_r^2=0$, the parabola degenerates into the half line $x=y$, $x\geqslant 0$, $y\geqslant 0$. As for the hyperbola, its asymptotes are the $x=K_1$ and $y=K_2$ lines, and its main parameter $K=K_1K_2-K_3$ tells us on which side of the asymptotes it lies. Fig. 1(a) shows an example of a 4 intersecting points configuration, corresponding to 4 distinct roots of P_A.

3. Topology of the root regions, K_1, K_2 fixed

Assume K_1 and K_2 are fixed, then the hyperbola asymptotes are fixed. The values of the two remaining parameters u_r^2 and K enable us to determine the number of intersecting points between the parabola and the hyperbola. From the possible relative positions of the parabola and the hyperbola, we can sketch the shape of the regions where there will be 0, 2, or 4 intersecting points (Fig. 2).

These regions are separated by 3 double root curves, one starting from point D, and two from point C. This comes from the fact that the discriminant of P_A is a third degree polynomial in K with three real roots for all K_1 and K_2

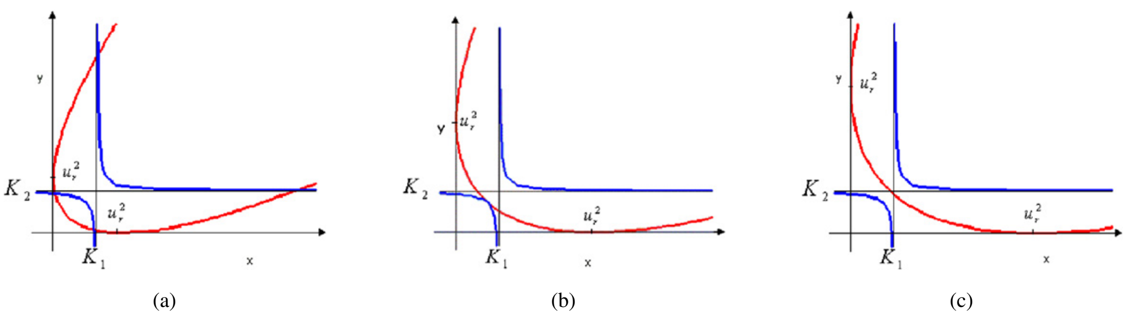

Fig. 1. 3 configurations: (a) four intersecting points configuration; (b) critical tangency configuration; (c) two intersecting points configuration.

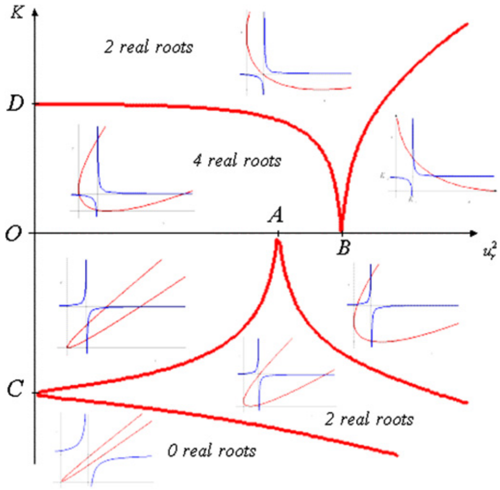

Fig. 2. Topology of P_A root regions, K_1, K_2 fixed.

and u_r^2. Though the equations of the double root curves depend on K_1 and K_2, the topology of the 0, 2, and 4 root regions is always the same. Besides, whatever the values of K_1 and K_2, the double root curves will meet the x axis at two points A and B, and the y axis at two other points C and D which correspond to special configurations of the hyperbola and the parabola.

3.1. *Coordinates of the key points A, B, C and D*

On the $K = 0$ axis, the hyperbola reduces to its asymptotes. The only way to obtain a double root is by having the parabola passing trough the hyperbola center. Writing the condition for (K_1, K_2) to be on the parabola, we obtain:

$$u_{rA}^2 = \left(\sqrt{K_1} - \sqrt{K_2}\right)^2, \qquad u_{rB}^2 = \left(\sqrt{K_1} + \sqrt{K_2}\right)^2. \tag{10}$$

On the $u_r^2 = 0$ axis, P_A reduces to the biquadratic function $X^4 - (K_1 + K_2)X^2 + K - K_1K_2$. Writing the condition for a double root to exist, we find C and D ordinates

$$K_D = K_1K_2, \qquad K_C = \left(\frac{K_1 - K_2}{2}\right)^2. \tag{11}$$

4. Hyperbolicity diagram

Considering from (6), that K_1, K_2, and $K = K_1K_2 - K_3$ are functions of Δp, the sketch of the root regions in a (K, u_r^2) frame on Fig. 2, can be transformed into a $(\Delta p, u_r^2)$ frame.

Proposition 4.1 *(Critical relative velocity). Whatever the value of* $\alpha_1, \alpha_2, \rho_1, \rho_2, c_1, c_2$ *in* $\mathbb{R}_+^*$, *there is a continuous strictly increasing function* $u_{rc}^2(\Delta p)$ *defined on* $[0, \rho_1 c_1^2]$ *with* $u_{rc}^2(0) = 0$, *and such that*

- *if* $u_r^2 \leqslant u_{rc}^2(\Delta p)$, *then* P_A *admits* 4 *real roots*;
- *if* $u_r^2 > u_{rc}^2(\Delta p)$, *then* P_A *admits only* 2 *real roots.*

Sketch of the proof. For any $K_1 \in \mathbb{R}_+^*$, and $K_2 \in \mathbb{R}_+^*$, consider the part of the double root curve on Fig. 2 going from D to B. For any $K \in [0, K_D]$, it corresponds to critical values $u_{rc}^2(K, K_1, K_2)$ such that the parabola is tangent to the lower branch of the hyperbola, as on Fig. 1(b). If $u_r^2 \leqslant u_{rc}^2(K, K_1, K_2)$ we are in a 4 real root region, Fig. 1(a), and if $u_{rc}^2(K, K_1, K_2) < u_r^2 \leqslant u_{rB}^2$ we are in a 2 real root region, Fig. 1(c). As the lower branch of the hyperbola moves down the origin with increasing values of K, u_{rc}^2 is a decreasing function of K. Moreover, the center of the hyperbola

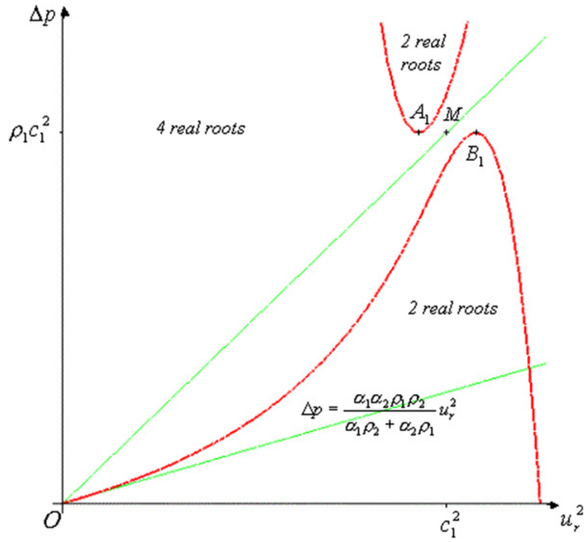

Fig. 3. Hyperbolicity diagram of the two-fluid model.

has coordinates (K_1, K_2), and thus u_{rc}^2 is an increasing function of K_1 and K_2. Now from (6), K_1 and K_2 are strictly increasing functions of Δp, and

$$K = \frac{\alpha_1\alpha_2\gamma^4}{c_1^2 c_2^2}\big(\Delta P - \rho_1 c_1^2\big)\big(\Delta P - \rho_2 c_2^2\big)$$

is a strictly decreasing function of $\Delta p \in [0, \rho_1 c_1^2]$. Therefore, $u_{rc}^2(K, K_1, K_2)$ is a strictly decreasing function of Δp in $[0, \rho_1 c_1^2]$. Moreover if $\Delta p = 0$ then $K = K_1 K_2$, and considering that whatever $K_1, K_2 \in \mathbb{R}_+^*$ we have $u_{rc}^2(K_1K_2, K_1, K_2) = 0$ (situation illustrated by point D on Fig. 2), we conclude that the starting point of the curve $u_{rc}^2(\Delta p)$ is the origin O. □

Fig. 3 represents the root regions of the two-fluid model (1)–(4) on a $(\Delta p, u_r^2)$ frame. We used the following numerical values corresponding to an air–water mixture at normal temperature and pressure: $\rho_1 = 1$, $\rho_2 = 1000$, $c_g = 340$, $c_l = 1500$ and $\alpha_1 = 0.7$. The line $\Delta p = \rho_1 c_1^2$ corresponds to the axis $K = 0$ on Fig. 2, which always belongs to the 4 real roots region. Points A and B transform into points A_1 and B_1, and from (10) we can determine their ordinates:

$$u_{rA_1}^2 = \big(c_1 - \gamma\sqrt{\rho_1}\big)^2, \qquad u_{rB_1}^2 = \big(c_1 + \gamma\sqrt{\rho_1}\big)^2. \tag{12}$$

4.1. Hyperbolic closure laws

A given model of interfacial pressure can be regarded as a coupling between Δp and u_r^2, which materialises as a curve on the hyperbolicity diagram (Fig. 3) that might cross a non-hyperbolic region. From the expression of the characteristic polynomial (5), we can derive a Taylor expansion of the double root curve that starts at O for small values of u_r^2:

$$\Delta p = \frac{\alpha_1\alpha_2\rho_1\rho_2}{\alpha_1\rho_2 + \alpha_2\rho_1} u_r^2 + \mathrm{O}\big(u_r^4\big).$$

Hence, for a linear closure law $\Delta p = s\, u_r^2$ to remain in the hyperbolic region, a necessary condition is

$$s \geqslant \frac{\alpha_1\alpha_2\rho_1\rho_2}{\alpha_1\rho_2 + \alpha_2\rho_1}.$$

This might not be sufficient when the curvature at O is positive as is the case on Fig. 3.

In order to build a line that passes through the neck between A_1 and B_1, we define M as the point with coordinates $(c_1^2, \rho_1 c_1^2)$. From (12), M is always left of B_1 and is right of A_1 provided

$$\alpha_1 \geqslant \frac{\rho_1(c_2^2 - c_1^2)}{\rho_2 c_2^2 - \rho_1 c_1^2}.$$

K. El Amine [3] however proved that for any α_1 in [0, 1], the closure law $\Delta p = \rho_1(u_1 - u_2)^2$ makes the isentropic two-fluid model hyperbolic as long as $|u_1 - u_2| \leqslant c_1$.

5. Conclusions

A geometric interpretation enabled us to represent the location and topology of the non-hyperbolic regions of the isentropic two-fluid model on a $(\Delta p, u_r^2)$ frame (Fig. 3). From this analysis, it is possible to build simple linear closure laws for the interfacial pressure that remain in the hyperbolic region such as the closure $\Delta p = \rho_1(u_1 - u_2)^2$. Other choices of hyperbolic closure laws with a more detailed analysis can be found in [4]. Working with the complete six equations model including the energy equations would bring the obvious additional eigenvalues u_g and u_l to the system (see [6]). The non-trivial eigenvalues would still be roots of a polynomial of the same form as P_A. The present study can be extended to take into account the virtual mass force effects (see [5] and [4]).

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