



## Calculus of Variations

# Derivation of a plate theory for incompressible materials

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### Abstract

We derive a two-dimensional model for elastic plates as a  $\Gamma$ -limit of three-dimensional nonlinear elasticity with the constraint of incompressibility. The energy density of the reduced problem describes plate bending, and is determined from the elastic moduli at the identity of the energy density of the three-dimensional problem. Without the constraint of incompressibility,  $\Gamma$ -convergence to a plate theory was first derived by Friesecke, James and Müller. The main difficulty in the present result is the construction of a recovery sequence which satisfies pointwise the nonlinear constraint of incompressibility. *To cite this article: S. Conti, G. Dolzmann, C. R. Acad. Sci. Paris, Ser. I 344 (2007).*

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### Résumé

**Dérivation de la théorie non-linéaire des plaques avec la contrainte de l'incompressibilité.** Nous dérivons un modèle bidimensionnel pour les plaques élastiques comme  $\Gamma$ -limite de la théorie de l'élasticité tridimensionnelle avec contrainte d'incompressibilité. La densité d'énergie du problème réduit est déterminée à partir des modules d'élastiques de la densité d'énergie tridimensionnelle à l'identité. Sans contrainte d'incompressibilité, Friesecke, James et Müller sont les premiers à avoir rigoureusement justifié le modèle de plaque en flexion par  $\Gamma$ -convergence. La difficulté principale de l'extension de ce résultat au cas incompressible réside dans la construction, afin d'établir l'inégalité de  $\Gamma$ -limsup, d'une suite de déformations satisfaisant la contrainte non-linéaire d'incompressibilité. *Pour citer cet article : S. Conti, G. Dolzmann, C. R. Acad. Sci. Paris, Ser. I 344 (2007).*

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In the past decade mathematical tools have been developed to derive in a rigorous way two-dimensional reduced models from the variational problem of three-dimensional nonlinear elasticity. The fundamental notion of convergence in this context is the concept of  $\Gamma$ -convergence for functionals as introduced by De Giorgi [7], see also [6,1].

A number of results for various theories including plate, shell, and membrane theories have been obtained in [12, 13,15,11,9,16,10] (see also the references therein). In all these results, the free energy density is assumed to be smooth on an open set in the space of deformation gradients. This assumption excludes for example the important class of rubber-like materials for which the energy density incorporates an incompressibility constraint, in the sense that the energy is infinite for all deformation gradients  $F$  that do not satisfy the nonlinear condition  $\det F = 1$ . A membrane

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theory for incompressible materials was derived independently by Trabelsi [17,18] and by the authors [5] and is relevant, for example, in the analysis of soft elasticity in thin sheets of nematic elastomers [19,8,3,20].

We focus here on the derivation of a plate theory for incompressible materials. A plate theory was first formulated by Kirchhoff back in 1850; more recent justifications are based on asymptotic expansion [2]. A derivation of Kirchhoff's plate theory without a priori assumptions was obtained in 2002 by Friesecke, James and Müller via  $\Gamma$ -convergence [11,9]. Their derivation holds for arbitrary energy densities, with quadratic growth from below, and  $C^2$ -smoothness in a neighborhood of the minimum. Under an additional a priori assumption on the smallness of the nonlinear strain the result was derived by Pantz [15,16]. In this paper, we address the incompressible case, where the energy density is finite and smooth only on a subset of the manifold of volume-preserving deformation gradients. Analogously to the membrane case, the main difficulty lies in the construction of a low-energy sequence of deformations which satisfy the incompressibility constraint pointwise.

We consider a thin elastic sheet of cross-section  $\omega \subset \mathbb{R}^2$ , and thickness  $h > 0$ . The nonlinear elastic energy is given by the functional  $I_h : W^{1,2}(\omega \times (0, h); \mathbb{R}^3) \rightarrow [0, \infty]$  defined through

$$I_h[u] = \frac{1}{h^3} \int_{\omega \times (0, h)} W(\nabla u) \, dx \, dx_3 \quad (1)$$

where we write  $x = (x_1, x_2)$  for the in-plane variables. In turn, the energy density  $W : \mathbb{M}^{3 \times 3} \rightarrow [0, \infty]$  is defined as

$$W(F) = \begin{cases} W_0(F) & \text{if } \det F = 1, \\ \infty & \text{else.} \end{cases} \quad (2)$$

Here  $W_0 : \mathbb{M}^{3 \times 3} \rightarrow [0, \infty]$  is frame indifferent, in the sense that  $W_0(QF) = W_0(F)$  for all  $Q \in \text{SO}(3)$ . Moreover  $W_0(F) = 0$  if  $F \in \text{SO}(3)$ ,  $W_0(F) > 0$  if  $F \notin \text{SO}(3)$ ,  $W_0$  is  $C^2$  smooth in a neighborhood of the  $3 \times 3$  identity matrix  $\text{Id}_3$ , and satisfies standard quadratic growth from below,

$$\frac{1}{c} |F|^2 - c \leq W_0(F), \quad \text{for some } c > 0. \quad (3)$$

The limiting theory describes only the deformation of the mid-plane of the sheet,  $v : \omega \rightarrow \mathbb{R}^3$ . Further, as usual in plate theories, the limiting functional is finite only on isometries, i.e., only on maps  $v$  such that  $\nabla v \in \text{O}(2, 3)$  a.e., where  $\text{O}(2, 3) = \{F \in \mathbb{M}^{3 \times 2} : F^T F = \text{Id}_2\}$ ,  $\text{Id}_2$  being the  $2 \times 2$  identity matrix.

The limiting energy of such an isometry is given by its bending energy, and depends on the second fundamental form  $\Pi_v : \omega \rightarrow \mathbb{M}^{2 \times 2}$ ,

$$\Pi_v = (\nabla v)^T \nabla b_v, \quad b_v = \partial_1 v \wedge \partial_2 v. \quad (4)$$

Here  $b_v$  is the normal to the surface described by  $v$  (notice that,  $v$  being an isometry,  $b_v$  is a unit vector).

The limiting energy can be expressed via the elastic coefficients of the three-dimensional energy density. Consider the Hessian of  $W_0$ ,

$$Q_3 = \nabla^2 W_0(\text{Id}_3), \quad (5)$$

which we view as a quadratic form on  $\mathbb{M}^{3 \times 3}$ . We show that the effective two-dimensional bending moduli are obtained by minimization of  $Q_3$  in the out-of-plane direction, subject to the (linearized) incompressibility constraint. Precisely, let  $Q_2$  be the quadratic form on  $\mathbb{M}^{2 \times 2}$  defined by

$$Q_2(G) = \min\{Q_3(G|d) : d \in \mathbb{R}^3, \text{Tr}(G|d) = 0\}, \quad (6)$$

where  $(G|d)$  is the  $3 \times 3$  matrix whose first two-by-two block is given by  $G \in \mathbb{M}^{2 \times 2}$ , the third column by  $d \in \mathbb{R}^3$ , and the remaining entries are zero.

We obtain that the limiting functional is given by

$$J[v] = \begin{cases} \frac{1}{24} \int_{\omega} Q_2(\Pi_v) \, dx & \text{if } v \in W^{2,2}(\omega; \mathbb{R}^3), \text{ with } \nabla v \in \text{O}(2, 3) \text{ a.e.,} \\ \infty & \text{else.} \end{cases} \quad (7)$$

The appropriate notion of convergence for deformation fields is, as usual for plates [9], strong convergence in  $W^{1,2}(\omega \times (0, 1); \mathbb{R}^3)$  of the rescaled deformations

$$U(x, x_3) = u(x, hx_3) \quad \text{for } x \in \omega \text{ and } x_3 \in (0, 1). \tag{8}$$

For the limiting map  $v : \omega \rightarrow \mathbb{R}^3$  the rescaling reduces to  $V(x, x_3) = v(x)$ .

**Theorem 1.** *Let  $\omega$  be a bounded, convex Lipschitz domain in  $\mathbb{R}^2$ , and suppose that  $W$  satisfies (2), (3). Then the following assertions hold:*

- (i) (Compactness) *For every sequence  $h_j \rightarrow 0$ , and every sequence  $u_j \in W^{1,2}(\omega \times (0, h_j); \mathbb{R}^3)$  such that  $I_{h_j}[u_j] < C < \infty$ , there exists a  $v \in W^{2,2}(\omega; \mathbb{R}^3)$ , with  $\nabla v \in O(2, 3)$  a.e., and a subsequence such that*

$$U_{j_k} - \frac{1}{|\omega|} \int_{\omega \times (0,1)} U_{j_k} \, dx \, dx_3 \rightarrow V \quad \text{in } W^{1,2}(\omega \times (0, 1); \mathbb{R}^3)$$

where  $U_j$  and  $V$  are related to  $u_j$  and  $v$  by (8).

- (ii) (Lower bound) *If the sequence  $u_j$  satisfies in addition  $U_j \rightarrow V$  in  $W^{1,2}$ , then*

$$\liminf_{j \rightarrow \infty} I_{h_j}[u_j] \geq J[v].$$

- (iii) (Upper bound) *For any  $v \in W^{2,2}(\omega; \mathbb{R}^3)$  with  $\nabla v \in O(2, 3)$  a.e., and any sequence  $h_j \rightarrow 0$  there is a sequence  $u_j \in C^\infty(\omega \times (0, h_j); \mathbb{R}^3)$  such that  $U_j \rightarrow V$  in  $W^{1,2}$  and*

$$\limsup_{j \rightarrow \infty} I_{h_j}[u_j] \leq J[v].$$

To prove (i) and (ii), we consider the energies

$$W^k(F) = W_0(F) + \frac{1}{2}k(\det F - 1)^2, \tag{9}$$

and let, analogously to (5) and (6),

$$Q_3^k(F) = \nabla^2 W^k(\text{Id}_3)(F, F) = Q_3(F) + k(\text{Tr } F)^2, \quad Q_2^k(G) = \min\{Q_3^k(G|d) : d \in \mathbb{R}^3\}. \tag{10}$$

The result of [9] and the fact that  $W^k \geq W$  imply compactness (and hence (i)) and that if  $U_j \rightarrow V$  then

$$\liminf_{j \rightarrow \infty} I_{h_j}[u_j] \geq \begin{cases} \frac{1}{24} \int_{\omega} Q_2^k(\Pi_v) \, dx & \text{if } \nabla v \in W^{1,2}(\omega; O(2, 3)), \\ \infty & \text{else.} \end{cases} \quad \text{for all } k > 0.$$

Taking  $k \rightarrow \infty$ , we have  $Q_2^k \rightarrow Q_2$ , and (ii) is proven.

To prove (iii) one needs to approximate  $v$  in energy by a sequence  $u_j$  such that  $\det \nabla u_j = 1$  everywhere. The first step is an approximation result by Pakzad [14]. He has shown that, on convex domains, any  $W^{2,2}$  isometry can be approximated, in  $W^{2,2}$ , by smooth isometries. Therefore it suffices to consider the case of smooth  $u$ . We define

$$u(x, x_3) = v(x) + \varphi(x, x_3)b_v(x) + \frac{1}{2}\varphi(x, x_3)^2d(x). \tag{11}$$

The function  $\varphi$  is chosen so that  $\varphi(x, 0) = 0$  and  $\det \nabla u = 1$  everywhere. For smooth  $v$  and  $d$ , this can be done on a domain of the form  $\omega \times (-\delta, \delta)$ , see [17,18,5]. The function  $d : \omega \rightarrow \mathbb{R}^3$  is chosen by a suitable smoothing of the optimal  $d$  entering (6), for  $G = \Pi_v$ . Finally,  $u_j(x, x_3) = u(x, x_3 - h_j/2)$ . The estimate of the energy is, in turn, analogous to the one in [9]. Details will be presented elsewhere [4].

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