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Partial Differential Equations

The biharmonic problem in the half-space with traces in weighted Sobolev spaces

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Abstract

In this Note, we study the biharmonic equation in the half-space $\mathbb{R}_+^N$, with $N \geqslant 2$. We prove in L^p theory, with $1 < p < \infty$, existence, uniqueness and regularity results; then we are interested in singular boundary conditions. We consider data and give solutions which live in weighted Sobolev spaces. ***To cite this article: C. Amrouche, Y. Raudin, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Le problème biharmonique dans le demi-espace avec traces dans des espaces de Sobolev avec poids. Nous étudions dans cet article l'équation biharmonique dans le demi-espace $\mathbb{R}_+^N$, avec $N \geqslant 2$. Nous donnons en théorie L^p, avec $1 < p < \infty$, des résultats d'existence, d'unicité et de régularité ; nous nous intéressons ensuite à des conditions aux limites singulières. Nous considèrons des données et fournissons des solutions dans des espaces de Sobolev avec poids. ***Pour citer cet article : C. Amrouche, Y. Raudin, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

L'objet de cette Note est de résoudre le problème biharmonique $(\mathcal{P})$ dans le demi-espace. Comme pour le problème de Poisson ou celui de Laplace dans un domaine extérieur, les espaces de Sobolev avec poids (1) fournissent un cadre très approprié pour la recherche de solutions. La principale différence est due à la nature de la frontière et l'une des difficultés est d'obtenir les espaces de traces qui conviennent (voir Lemme 1.1). Les résultats principaux sont contenus dans le Théorème 2.1 qui donne l'existence de solutions généralisées, le Théorème 3.1 pour les solutions dites faibles, intermédiaires aux généralisées et aux solutions fortes données dans le Théorème 3.2. Par un argument de dualité, nous étudions aussi l'existence de solutions très faibles correspondants à des données au bord singulières.

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1. Introduction

The purpose of this Note is the resolution of the biharmonic problem with nonhomogeneous boundary conditions in the half-space

$$(\mathcal{P}):\quad \Delta^2 u = f \quad \text{in } \mathbb{R}^N_+, \qquad u = g_0 \quad \text{and} \quad \partial_N u = g_1 \quad \text{on } \Gamma = \mathbb{R}^{N-1}.$$

Since this problem is posed in the half-space, it is important to specify the behavior at infinity for the data and solutions. We have chosen to impose such conditions by setting our problem in weighted Sobolev spaces which provide a correct functional setting for unbounded domains. Our analysis is based on the isomorphism properties of the biharmonic operator in the whole space and the resolution of the Dirichlet and Neumann problems for the Laplacian in the half-space and also on the reflection principle inherent in the half-space. We begin to establish the existence of generalized solutions to problem $(\mathcal{P})$, i.e. solutions which belong to weighted Sobolev spaces of type $W^{2,p}_{\ell}(\mathbb{R}^N_+)$. Then we are interested both in the existence of strong solutions which belong to spaces of type $W^{4,p}_{\ell+2}(\mathbb{R}^N_+)$, and singular solutions which belong to $W^{0,p}_{\ell-2}(\mathbb{R}^N_+)$ or $W^{1,p}_{\ell-1}(\mathbb{R}^N_+)$ with singular boundary conditions. We also establish the existence of solutions which belong to the intermediate spaces $W^{3,p}_{\ell+1}(\mathbb{R}^N_+)$. The detailed proofs of the results announced in this Note are given in [2] and in a forthcoming paper.

It turns out that the use of classical Sobolev spaces is inadequate in this case, contrary to the study of elliptic problems of type:

$$(\mathcal{Q}):\quad u + \Delta^2 u = f \quad \text{in } \mathbb{R}^N_+, \qquad u = g_0 \quad \text{and} \quad \partial_N u = g_1 \quad \text{on } \Gamma,$$

where it is more reasonable to consider data and solutions in standard Sobolev spaces. For example, if $f \in L^2(\mathbb{R}^N_+)$, $g_0 \in H^{7/2}(\mathbb{R}^{N-1})$ and $g_1 \in H^{5/2}(\mathbb{R}^{N-1})$, problem $(\mathcal{Q})$ admits a unique solution $u \in H^4(\mathbb{R}^N_+)$. In the case of problem $(\mathcal{P})$, if we assume that $f \in L^2(\mathbb{R}^N_+)$, the solution u cannot be better than in $W^{4,2}_0(\mathbb{R}^N_+)$ and its traces $u|_\Gamma$ and $\partial_N u|_\Gamma$ respectively in $W^{7/2,2}_0(\mathbb{R}^{N-1})$ and $W^{5/2,2}_0(\mathbb{R}^{N-1})$. Moreover, we can observe that these spaces are respective subspaces of the first.

On the one hand, we can find in the literature an approach via homogeneous spaces. For instance, when $f \in L^2(\mathbb{R}^N_+)$, that consists in finding solutions to $(\mathcal{P})$ satisfying $\partial^4 u \in L^2(\mathbb{R}^N_+)$. But that gives no information on the other derivatives, nor specifies the behavior at infinity for the data and solutions. On the other hand, Boulmezaoud has established (see [3]) in a Hilbertian framework, the existence of solutions $u \in W^{3,2}_{\ell+1}(\mathbb{R}^N_+)$ for data $f \in W^{-1,2}_{\ell+1}(\mathbb{R}^N_+)$ and regularity results. However, owing to some critical cases, this framework excludes in particular the dimensions 2 and 4.

For any integer $N \geqslant 2$, writing a typical point of $\mathbb{R}^N$ as $x = (x', x_N)$, we denote by $\mathbb{R}^N_+$ the upper half-space of $\mathbb{R}^N$ and Γ its boundary. We shall use the two basic weights $\varrho = (1 + |x|^2)^{1/2}$ and $\lg \varrho = \ln(2 + |x|^2)$, where $|x|$ is the Euclidean norm of x. For any integer q, $\mathcal{P}_q$ stands for the space of polynomials of degree smaller than or equal to q; $\mathcal{P}^{\Delta}_q$ (resp. $\mathcal{P}^{\Delta^2}_q$) is the subspace of harmonic (resp. biharmonic) polynomials of $\mathcal{P}_q$; $\mathcal{A}^{\Delta}_q$ (resp. $\mathcal{N}^{\Delta}_q$) is the subspace of polynomials of $\mathcal{P}^{\Delta}_q$, odd (resp. even) with respect to x_N, or equivalently, which satisfy the condition $\varphi(x', 0) = 0$ (resp. $\partial_N \varphi(x', 0) = 0$); with the convention that these spaces are reduced to $\{0\}$ if $q < 0$. For any real number s, we denote by $[s]$ the integer part of s. Given a Banach space B, with dual space B' and a closed subspace X of B, we denote by $B' \perp X$ the subspace of B' orthogonal to X. For any $k \in \mathbb{Z}$, we shall denote by $\{1, \dots, k\}$ the set of the first k positive integers, with the convention that this set is empty if k is nonpositive.

Let Ω be an open set of $\mathbb{R}^N$. For any $m \in \mathbb{N}$, $p \in \,]1, \infty[$, $(\alpha, \beta) \in \mathbb{R}^2$, we define the following space:

$$\begin{aligned} W^{m,p}_{\alpha,\beta}(\Omega) = \big\{ u \in \mathcal{D}'(\Omega);\ & 0 \leqslant |\lambda| \leqslant k,\ \varrho^{\alpha - m + |\lambda|}(\lg \varrho)^{\beta - 1} \partial^\lambda u \in L^p(\Omega); \\ & k + 1 \leqslant |\lambda| \leqslant m,\ \varrho^{\alpha - m + |\lambda|}(\lg \varrho)^{\beta} \partial^\lambda u \in L^p(\Omega) \big\}, \end{aligned} \tag{1}$$

where $k = m - N/p - \alpha$ if $N/p + \alpha \in \{1, \dots, m\}$, and $k = -1$ otherwise. In the case $\beta = 0$, we simply denote the space by $W^{m,p}_{\alpha}(\Omega)$. Note that $W^{m,p}_{\alpha,\beta}(\Omega)$ is a reflexive Banach space equipped with its graph norm. Now, we define the space

$$\mathring{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) = \overline{\mathcal{D}(\mathbb{R}^N_+)}^{\|\cdot\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}},$$

and its dual is denoted by $W^{-m,p'}_{-\alpha,-\beta}(\mathbb{R}^N_+)$. In order to define the traces of functions of $W^{m,p}_{\alpha}(\mathbb{R}^N_+)$ (here we do not consider the case $\beta \neq 0$), for any $\sigma \in]0,1[$, we introduce the space:

$$W^{\sigma,p}_{\alpha}(\mathbb{R}^N) = \left\{ u \in \mathcal{D}'(\mathbb{R}^N);\ w^{\alpha-\sigma} u \in L^p(\mathbb{R}^N),\ \int\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varrho^{\alpha}(x)u(x) - \varrho^{\alpha}(y)u(y)|^p}{|x-y|^{N+\sigma p}}\, \mathrm{d}x\, \mathrm{d}y < \infty \right\},$$

where $w = \varrho$ if $N/p + \alpha \neq \sigma$ and $w = \varrho(\lg \varrho)^{1/(\sigma-\alpha)}$ if $N/p + \alpha = \sigma$. For any $s \in \mathbb{R}^+$, we set

$$\begin{aligned} W^{s,p}_{\alpha}(\mathbb{R}^N) = \big\{ u \in \mathcal{D}'(\mathbb{R}^N);\ & 0 \leqslant |\lambda| \leqslant k,\ \varrho^{\alpha-s+|\lambda|}(\lg \varrho)^{-1} \partial^{\lambda} u \in L^p(\mathbb{R}^N); \\ & k+1 \leqslant |\lambda| \leqslant [s]-1,\ \varrho^{\alpha-s+|\lambda|} \partial^{\lambda} u \in L^p(\mathbb{R}^N);\ \partial^{[s]} u \in W^{\sigma,p}_{\alpha}(\mathbb{R}^N) \big\}, \end{aligned}$$

where $k = s - N/p - \alpha$ if $N/p + \alpha \in \{\sigma, \ldots, \sigma + [s]\}$, with $\sigma = s - [s]$ and $k = -1$ otherwise. We also define, for any real number β, the space $W^{s,p}_{\alpha,\beta}(\mathbb{R}^N) = \{ v \in \mathcal{D}'(\mathbb{R}^N);\ (\lg \varrho)^{\beta} v \in W^{s,p}_{\alpha}(\mathbb{R}^N)\}$.

Let us recall, for any integer $m \geqslant 1$ and any real number α, the following trace lemma:

Lemma 1.1. *The following linear mapping is continuous and surjective*:

$$\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{m-1}) : W^{m,p}_{\alpha}(\mathbb{R}^N_+) \longrightarrow \prod_{j=0}^{m-1} W^{m-j-1/p,p}_{\alpha}(\mathbb{R}^{N-1}).$$

2. Generalized solutions of Δ^2 in $\mathbb{R}^N_+$

Before to deal with the biharmonic problem in the half-space, we need to establish isomorphism results for the biharmonic operator in $\mathbb{R}^N$. Let $\ell \in \mathbb{Z}$, combining isomorphism results for Δ in $\mathbb{R}^N$ (see [1]), we can show that the following operators are isomorphisms

$$\Delta^2 : W^{2,p}_{\ell}(\mathbb{R}^N)/\mathcal{P}^{\Delta^2}_{[2-\ell-N/p]} \longrightarrow W^{-2,p}_{\ell}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta^2}_{[2+\ell-N/p']}, \tag{2}$$

$$\Delta^2 : W^{3,p}_{\ell+1}(\mathbb{R}^N)/\mathcal{P}^{\Delta^2}_{[2-\ell-N/p]} \longrightarrow W^{-1,p}_{\ell+1}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta^2}_{[2+\ell-N/p']}, \tag{3}$$

$$\Delta^2 : W^{4,p}_{\ell+2}(\mathbb{R}^N)/\mathcal{P}^{\Delta^2}_{[2-\ell-N/p]} \longrightarrow W^{0,p}_{\ell+2}(\mathbb{R}^N) \perp \mathcal{P}^{\Delta^2}_{[2+\ell-N/p']}, \tag{4}$$

respectively under the following hypotheses

$$N/p' \notin \{1, \ldots, \ell\} \quad \text{and} \quad N/p \notin \{1, \ldots, -\ell\}, \tag{5}$$

$$N/p' \notin \{1, \ldots, \ell+1\} \quad \text{and} \quad N/p \notin \{1, \ldots, -\ell-1\}, \tag{6}$$

$$N/p' \notin \{1, \ldots, \ell+2\} \quad \text{and} \quad N/p \notin \{1, \ldots, -\ell-2\}. \tag{7}$$

For any $q \in \mathbb{Z}$, we introduce the space $\mathcal{B}_q = \{u \in \mathcal{P}^{\Delta^2}_q;\ u = \partial_N u = 0 \text{ on } \Gamma\}$.

Theorem 2.1 *(Generalized solutions). Let $\ell \in \mathbb{Z}$. Under hypothesis* (5)*, for any $f \in W^{-2,p}_{\ell}(\mathbb{R}^N_+)$, $g_0 \in W^{2-1/p,p}_{\ell}(\Gamma)$ and $g_1 \in W^{1-1/p,p}_{\ell}(\Gamma)$ satisfying the compatibility condition*

$$\forall \varphi \in \mathcal{B}_{[2+\ell-N/p']}, \quad \langle f, \varphi \rangle_{W^{-2,p}_{\ell}(\mathbb{R}^N_+) \times \mathring{W}^{2,p'}_{-\ell}(\mathbb{R}^N_+)} + \langle g_1, \Delta\varphi \rangle_{\Gamma} - \langle g_0, \partial_N \Delta\varphi \rangle_{\Gamma} = 0, \tag{8}$$

problem $(\mathcal{P})$ *admits a solution $u \in W^{2,p}_{\ell}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{B}_{[2-\ell-N/p]}$, and there exists a constant C such that*

$$\inf_{q \in \mathcal{B}_{[2-\ell-N/p]}} \|u+q\|_{W^{2,p}_{\ell}(\mathbb{R}^N_+)} \leqslant C\big(\|f\|_{W^{-2,p}_{\ell}(\mathbb{R}^N_+)} + \|g_0\|_{W^{2-1/p,p}_{\ell}(\Gamma)} + \|g_1\|_{W^{1-1/p,p}_{\ell}(\Gamma)}\big).$$

Proof. (i) We begin to characterize the kernel $\mathcal{K}$ of the operator $(\Delta^2, \gamma_0, \gamma_1)$ in $W^{2,p}_{\ell}(\mathbb{R}^N_+)$. Thanks to the reflection principle for the biharmonic equation, we show that if $\ell \in \mathbb{Z}$, assuming that $N/p \notin \{1, \ldots, -\ell\}$, then

$\mathcal{K} = \mathcal{B}_{[2-\ell-N/p]}$. Moreover, we define the two operators Π_D and Π_N, respectively for any $r \in \mathcal{A}_k^{\Delta}$ and $s \in \mathcal{N}_k^{\Delta}$, by

$$\Pi_D r(x) = \frac{1}{2}\int_0^{x_N} \mathrm{tr}(x',t)\,\mathrm{d}t \quad \text{and} \quad \Pi_N s(x) = \frac{1}{2}x_N \int_0^{x_N} s(x',t)\,\mathrm{d}t,$$

satisfying the following properties. For any $r \in \mathcal{A}_k^{\Delta}$, we have $\Delta \Pi_D r = r$ in $\mathbb{R}_+^N$ and $\Pi_D r = \partial_N \Pi_D r = 0$ on Γ and similar relations for any $s \in \mathcal{N}_k^{\Delta}$, with the operator Π_N. Assuming that $N/p \notin \{1, \dots, -\ell\}$, then we prove that

$$\mathcal{K} = \mathcal{B}_{[2-\ell-N/p]} = \Pi_D \mathcal{A}^{\Delta}_{[-\ell-N/p]} \oplus \Pi_N \mathcal{N}^{\Delta}_{[-\ell-N/p]}. \tag{9}$$

Using a Green formula and the density of $\mathcal{D}(\overline{\mathbb{R}_+^N})$ in $W_\ell^{2,p}(\mathbb{R}_+^N)$, we can easily prove the necessity of the compatibility condition (8).

(ii) We establish the result in the case $f = 0$. Thanks to the relation (9), the compatibility condition (8) is equivalent to both conditions: $\forall r \in \mathcal{A}^{\Delta}_{[\ell-N/p']}$, $\langle g_0, \partial_N r\rangle_\Gamma = 0$ and $\forall s \in \mathcal{N}^{\Delta}_{[\ell-N/p']}$, $\langle g_1, s\rangle_\Gamma = 0$. These conditions assure the existence of $\vartheta \in W_\ell^{2,p}(\mathbb{R}_+^N)$ such that $\Delta\vartheta = 0$ in $\mathbb{R}_+^N$, $\vartheta = g_0$ on Γ and of $\zeta \in W_\ell^{2,p}(\mathbb{R}_+^N)$ such that $\Delta\zeta = 0$ in $\mathbb{R}_+^N$, $\zeta = g_1$ on Γ. We can readily verify that the function defined by $u = x_N\partial_N(\zeta - \vartheta) + \vartheta \in W_{\ell-1}^{1,p}(\mathbb{R}_+^N)$ verifies problem $(\mathcal{P})$. Furthermore, we remark that u also satisfies $\Delta u = 2\partial_N^2(\zeta - \vartheta)$ in $\mathbb{R}_+^N$, $u = g_0$ on Γ and by a uniqueness argument, we can deduce that $u \in W_\ell^{2,p}(\mathbb{R}_+^N)$.

(iii) In the general case, thanks to Lemma 1.1, we consider a lifting function $u_g \in W_\ell^{2,p}(\mathbb{R}_+^N)$ of (g_0, g_1), so that to solve $(\mathcal{P})$ is equivalent solving

$$(\mathcal{P}^\star):\quad \Delta^2 u = f \quad \text{in } \mathbb{R}_+^N, \qquad u = \partial_N u = 0 \quad \text{on } \Gamma,$$

where $f \in W_\ell^{-2,p}(\mathbb{R}_+^N)$ with the orthogonality condition $f \perp \mathcal{B}_{[2+\ell-N/p']}$.

Now, thanks to Hardy inequality (see [1]), we can write $f = \operatorname{div}\operatorname{div}\mathbb{F}$, where $\mathbb{F} \in W_\ell^{0,p}(\mathbb{R}_+^N)^{N^2}$.

step 1. Assume that $2+\ell-N/p' < 0$. Let $\tilde{\mathbb{F}}$ be the extension of $\mathbb{F}$ to $\mathbb{R}^N$ by 0 and $\tilde f = \operatorname{div}\operatorname{div}\tilde{\mathbb{F}} \in W_\ell^{-2,p}(\mathbb{R}^N)$. By isomorphism (2), there exists $\tilde z \in W_\ell^{2,p}(\mathbb{R}^N)$ such that $\tilde f = \Delta^2 \tilde z$ in $\mathbb{R}^N$. Setting $z = \tilde z|_{\mathbb{R}_+^N}$, since $\mathcal{B}_{[2+\ell-N/p']} = \{0\}$, there exists $v \in W_\ell^{2,p}(\mathbb{R}_+^N)$ such that

$$(\mathcal{P}^\sharp):\quad \Delta^2 v = 0 \quad \text{in } \mathbb{R}_+^N, \qquad v = z \quad \text{and} \quad \partial_N v = \partial_N z \quad \text{on } \Gamma.$$

The function $u = z - v$ answers to problem $(\mathcal{P}^\star)$ in this case.

step 2. Assume that $2-\ell-N/p < 0$. We have shown in the case $2+\ell-N/p' < 0$, that the operator Δ^2 is an isomorphism from $\mathring{W}_\ell^{2,p}(\mathbb{R}_+^N)/\mathcal{B}_{[2-\ell-N/p]}$ onto $W_\ell^{-2,p}(\mathbb{R}_+^N)$. Thus, if $2-\ell-N/p < 0$, we can deduce by duality that the operator Δ^2 is also an isomorphism from $\mathring{W}_\ell^{2,p}(\mathbb{R}_+^N)$ onto $W_\ell^{-2,p}(\mathbb{R}_+^N) \perp \mathcal{B}_{[2+\ell-N/p']}$.

step 3. Assume that $2+\ell-N/p' \geqslant 0$ and $2-\ell-N/p \geqslant 0$, which implies $\ell \in \{-1,0,1\}$. If $\ell \in \{-1,0\}$, we use again the extension $\tilde{\mathbb{F}}$ of $\mathbb{F}$ to $\mathbb{R}^N$ by 0. By isomorphism (4), there exists $\tilde{\mathbb{G}} \in W_\ell^{4,p}(\mathbb{R}^N)^{N^2}$ such that $\Delta^2\tilde{\mathbb{G}} = \tilde{\mathbb{F}}$ in $\mathbb{R}^N$. Setting $\mathbb{G} = \tilde{\mathbb{G}}|_{\mathbb{R}_+^N}$ and $z = \operatorname{div}\operatorname{div}\mathbb{G}$, we obtain $z \in W_\ell^{2,p}(\mathbb{R}_+^N)$ and $\Delta^2 z = f$ in $\mathbb{R}_+^N$. Point (ii) yields a solution $v \in W_\ell^{2,p}(\mathbb{R}_+^N)$ to problem $(\mathcal{P}^\sharp)$. Then the function $u = z - v$ answers to problem $(\mathcal{P}^\star)$ for $\ell \in \{-1,0\}$. Now we can deduce the case $\ell = 1$ by duality from $\ell = -1$.

It remains to combine the three steps to obtain the isomorphism Δ^2 from $\mathring{W}_\ell^{2,p}(\mathbb{R}_+^N)/\mathcal{B}_{[2-\ell-N/p]}$ onto $W_\ell^{-2,p}(\mathbb{R}_+^N) \perp \mathcal{B}_{[2+\ell-N/p']}$, for any $\ell \in \mathbb{Z}$ verifying (5). This answers globally to problem $(\mathcal{P}^\star)$ and thus to general problem $(\mathcal{P})$. $\square$

3. Weak and strong solutions of Δ^2 in $\mathbb{R}_+^N$

Here, we will establish the first step of a regularity result:

Theorem 3.1 *(Weak solutions). Let $\ell \in \mathbb{Z}$. Under hypothesis* (6)*, for any $f \in W^{-1,p}_{\ell+1}(\mathbb{R}^N_+)$, $g_0 \in W^{3-1/p,p}_{\ell+1}(\Gamma)$ and $g_1 \in W^{2-1/p,p}_{\ell+1}(\Gamma)$ satisfying the compatibility condition* (8)*, problem* ($\mathcal{P}$) *admits a solution $u \in W^{3,p}_{\ell+1}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{B}_{[2-\ell-N/p]}$, and which continuously depends on the data with respect to the quotient norm.*

Proof. (i) We consider a lifting function $u_g \in W^{3,p}_{\ell+1}(\mathbb{R}^N_+)$ of (g_0, g_1), so that to solve ($\mathcal{P}$) is equivalent solving ($\mathcal{P}^\star$), where $f \in W^{-1,p}_{\ell+1}(\mathbb{R}^N_+)$ with the orthogonality condition $f \perp \mathcal{B}_{[3+\ell-N/p']}$.

(ii) Assume that $\ell \leqslant -2$. The idea consists to an extension $\tilde{f} \in W^{-1,p}_{\ell+1}(\mathbb{R}^N)$ of f and then successively to a resolution of a biharmonic problem in the whole space and a homogeneous problem ($\mathcal{P}^0$).

(iii) Assume that $\ell \geqslant -1$. As in [3], we begin to define from f the function $\xi_0 \in W^{1-1/p,p}_{\ell+1}(\Gamma)$ such that the problem $\Delta\xi = f$ in $\mathbb{R}^N_+$, $\xi = \xi_0$ on Γ admits a solution $\xi \in W^{1,p}_{\ell+1}(\mathbb{R}^N_+)$. Then there exists $\vartheta \in W^{3,p}_{\ell+1}(\mathbb{R}^N_+)$ such that $\Delta\vartheta = \xi$ in $\mathbb{R}^N_+$ and $\vartheta = 0$ on Γ and $\zeta \in W^{3,p}_{\ell+1}(\mathbb{R}^N_+)$ such that $\Delta\zeta = \xi$ in $\mathbb{R}^N_+$ and $\partial_N\zeta = 0$ on Γ. The function defined by $u = x_N\partial_N(\zeta - \vartheta) + \vartheta \in W^{2,p}_{\ell}(\mathbb{R}^N_+)$ verifies problem ($\mathcal{P}^\star$). Furthermore, we remark that u also satisfies $\Delta u = 2\partial^2_N(\zeta - \vartheta) + \xi$ in $\mathbb{R}^N_+$ and $u = 0$ on Γ. By a uniqueness argument, we can then deduce that $u \in W^{3,p}_{\ell+1}(\mathbb{R}^N_+)$. □

By a regularity argument, we can obtain the second step of this regularity result:

Theorem 3.2 *(Strong solutions). Let $\ell \in \mathbb{Z}$. Under hypothesis* (7)*, for any $f \in W^{0,p}_{\ell+2}(\mathbb{R}^N_+)$, $g_0 \in W^{4-1/p,p}_{\ell+2}(\Gamma)$ and $g_1 \in W^{3-1/p,p}_{\ell+2}(\Gamma)$ satisfying the compatibility condition* (8)*, problem* ($\mathcal{P}$) *admits a solution $u \in W^{4,p}_{\ell+2}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{B}_{[2-\ell-N/p]}$, and which continuously depends on the data with respect to the quotient norm.*

4. Very weak solutions and singular boundary conditions

To finish this study, we now come back to the homogeneous problem

$$(\mathcal{P}^0):\quad \Delta^2 u = 0 \quad \text{in } \mathbb{R}^N_+, \qquad u = g_0 \quad \text{and} \quad \partial_N u = g_1 \quad \text{on } \Gamma,$$

where we envisage singular boundary conditions g_0 and g_1.

Theorem 4.1 *(Very weak solutions). Let $\ell \in \mathbb{Z}$ and assume that*

$$N/p' \notin \{1, \dots, \ell - 2\} \quad \text{and} \quad N/p \notin \{1, \dots, -\ell + 2\}. \tag{10}$$

For any $g_0 \in W^{-1/p,p}_{\ell-2}(\Gamma)$ and $g_1 \in W^{-1-1/p,p}_{\ell-2}(\Gamma)$ satisfying the compatibility condition

$$\forall\varphi \in \mathcal{B}_{[2+\ell-N/p']}, \quad \langle g_1, \Delta\varphi\rangle_\Gamma - \langle g_0, \partial_N\Delta\varphi\rangle_\Gamma = 0, \tag{11}$$

problem ($\mathcal{P}^0$) *has a solution $u \in W^{0,p}_{\ell-2}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{B}_{[2-\ell-N/p]}$, and which continuously depends on the data with respect to the quotient norm.*

Proof. (i) Let $\ell \in \mathbb{Z}$ and $m \geqslant 2$ be two integers. We introduce the spaces

$$\overset{\star}{W}{}^{m,p}_{\ell}(\mathbb{R}^N_+) = \{u \in W^{m,p}_{\ell}(\mathbb{R}^N_+);\ u = \partial_N u = 0 \text{ on } \Gamma\}, \quad Y^p_{\ell,1}(\mathbb{R}^N_+) = \{v \in W^{0,p}_{\ell-2}(\mathbb{R}^N_+);\ \Delta^2 v \in W^{0,p}_{\ell+2,1}(\mathbb{R}^N_+)\}.$$

Note that $Y^p_{\ell,1}(\mathbb{R}^N_+)$ is a reflexive Banach space equipped with its graph norm. Then we can show that under hypothesis (10), the space $\mathcal{D}(\overline{\mathbb{R}^N_+})$ is dense in $Y^p_{\ell,1}(\mathbb{R}^N_+)$ and we can define a linear continuous mapping (γ_0, γ_1) from $Y^p_{\ell,1}(\mathbb{R}^N_+)$ into $W^{-1/p,p}_{\ell-2}(\Gamma) \times W^{-1-1/p,p}_{\ell-2}(\Gamma)$; and moreover we have for any $v \in Y^p_{\ell,1}(\mathbb{R}^N_+)$ and $\varphi \in \overset{\star}{W}{}^{4,p'}_{-\ell+2}(\mathbb{R}^N_+)$,

$$\begin{aligned} &\langle \Delta^2 v, \varphi\rangle_{W^{0,p}_{\ell+2,1}(\mathbb{R}^N_+)\times W^{0,p'}_{-\ell-2,-1}(\mathbb{R}^N_+)} - \langle v, \Delta^2\varphi\rangle_{W^{0,p}_{\ell-2}(\mathbb{R}^N_+)\times W^{0,p'}_{-\ell+2}(\mathbb{R}^N_+)} \\ &\quad = \langle v, \partial_N\Delta\varphi\rangle_{W^{-1/p,p}_{\ell-2}(\Gamma)\times W^{1/p,p'}_{-\ell+2}(\Gamma)} - \langle \partial_N v, \Delta\varphi\rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma)\times W^{1+1/p,p'}_{-\ell+2}(\Gamma)}. \end{aligned} \tag{12}$$

(ii) Let $\mathcal{K}^\flat = \{z \in W^{0,p}_{\ell-2}(\mathbb{R}^N_+);\ \Delta^2 z = 0 \text{ in } \mathbb{R}^N_+,\ z = \partial_N z = 0 \text{ on } \Gamma\}$ be the kernel of this operator. Thanks to the formula (12), we can observe that problem $(\mathcal{P}^0)$ is equivalent to the formulation:

$$(\mathcal{R})\quad \begin{cases} \text{Find } u \in Y^p_{\ell,1}(\mathbb{R}^N_+)/\mathcal{K}^\flat \text{ such that for any } v \in \overset{\star}{W}{}^{4,p'}_{-\ell+2}(\mathbb{R}^N_+), \\ \langle u, \Delta^2 v\rangle_{W^{0,p}_{\ell-2}(\mathbb{R}^N_+)\times W^{0,p'}_{-\ell+2}(\mathbb{R}^N_+)} = \langle g_1, \Delta v\rangle_\Gamma - \langle g_0, \partial_N \Delta v\rangle_\Gamma. \end{cases}$$

(iii) We shall solve problem $(\mathcal{R})$ by duality. According to Theorem 3.2, problem $(\mathcal{P}^\star)$ admits a unique solution $v \in W^{4,p'}_{-\ell+2}(\mathbb{R}^N_+)/\mathcal{B}_{[2+\ell-N/p']}$, under hypothesis (10). Moreover, v satisfies the estimate

$$\|v\|_{W^{4,p'}_{-\ell+2}(\mathbb{R}^N_+)/\mathcal{B}_{[2+\ell-N/p']}} \leqslant C\|f\|_{W^{0,p'}_{-\ell+2}(\mathbb{R}^N_+)}.$$

Consider the linear form $T : f \mapsto \langle g_1, \Delta v\rangle_\Gamma - \langle g_0, \partial_N \Delta v\rangle_\Gamma$. We can show that it is continuous on $W^{0,p'}_{-\ell+2}(\mathbb{R}^N_+) \perp \mathcal{B}_{[2-\ell-N/p]}$. Then, according to Riesz representation theorem, there exists a unique $u \in W^{0,p}_{\ell-2}(\mathbb{R}^N_+)/\mathcal{B}_{[2-\ell-N/p]}$ such that $Tf = \langle u, f\rangle_{W^{0,p}_{\ell-2}(\mathbb{R}^N_+)\times W^{0,p'}_{-\ell+2}(\mathbb{R}^N_+)}$. This means that u is a solution to problem $(\mathcal{R})$ and $\mathcal{K}^\flat = \mathcal{B}_{[2-\ell-N/p]}$. □

We can complete these results by an intermediate case, with a proof similar to the previous one:

Theorem 4.2. *Let $\ell \in \mathbb{Z}$ and assume that $N/p' \notin \{1, \ldots, \ell-1\}$ and $N/p \notin \{1, \ldots, -\ell+1\}$. For any $g_0 \in W^{1-1/p,p}_{\ell-1}(\Gamma)$ and $g_1 \in W^{-1/p,p}_{\ell-1}(\Gamma)$ satisfying the compatibility condition* (11)*, problem $(\mathcal{P}^0)$ has a solution $u \in W^{1,p}_{\ell-1}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{B}_{[2-\ell-N/p]}$, and which continuously depends on the data with respect to the quotient norm.*

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