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Extremes of the mass distribution associated with a trivariate quasi-copula

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Abstract

We identify the extremes of the mass distribution associated with a trivariate quasi-copula and compare our findings with the bivariate case. ***To cite this article: B. De Baets et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Extrêmes de la distribution de masse associée à une quasi-copula dans un espace tridimensionnel. Nous identifions les extrêmes de la distribution de masse associée à une quasi-copula dans un espace tridimensionnel. Les résultats sont comparés à ceux obtenus dans le cas bidimensionnel. ***Pour citer cet article : B. De Baets et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

1. Introduction

Alsina et al. [1] introduced the notion of a *quasi-copula* in order to characterize operations on distribution functions that can, or cannot, be derived from operations on random variables defined on the same probability space. Cuculescu and Theodorescu [3] characterize a multivariate quasi-copula – or n-quasi-copula, with $n \geqslant 2$ a natural number – as a function $Q:[0,1]^n \to [0,1]$ that satisfies:

(i) *boundary conditions*: $Q(u_1,\dots,u_{i-1},0,u_{i+1},\dots,u_n)=0$ and $Q(1,\dots,1,u_i,1,\dots,1)=u_i$, for any $(u_1,\dots,u_n)\in[0,1]^n$;
(ii) *monotonicity*: Q is increasing in each variable;
(iii) *Lipschitz condition*: $|Q(u_1,\dots,u_n)-Q(v_1,\dots,v_n)|\leqslant\sum_{i=1}^n|u_i-v_i|$, for any $(u_1,\dots,u_n)$ and $(v_1,\dots,v_n)$ in $[0,1]^n$.

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Every n-quasi-copula Q satisfies the following inequalities:

$$W^n(u_1,\dots,u_n)=\max\left(\sum_{i=1}^{n}u_i-n+1,0\right)\leqslant Q(u_1,\dots,u_n)\leqslant \min(u_1,\dots,u_n).$$

W^2 is a 2-copula – a stronger concept than that of a quasi-copula [5] – and W^n, $n\geqslant 3$, is a *proper* n-quasi-copula, i.e. an n-quasi-copula but not an n-copula.

Consider an n-quasi-copula Q and an n-box $B=[a_1,b_1]\times\cdots\times[a_n,b_n]$ in $[0,1]^n$. The *Q-volume* of B is defined as

$$V_Q(B)=\sum \operatorname{sgn}(c_1,\dots,c_n)Q(c_1,\dots,c_n),$$

where the sum is taken over all the *vertices* $(c_1,\dots,c_n)$ of B — i.e., each c_k is equal to either a_k or b_k — and $\operatorname{sgn}(c_1,\dots,c_n)$ is 1 if $c_k=a_k$ for an even number of k's, and -1 if $c_k=a_k$ for an odd number of k's. We refer to V_Q as the *mass distribution* associated with Q (on n-boxes), and to $V_Q(B)$ as the mass accumulated by Q on B.

In view of its usefulness for illustrating the differences between copulas and quasi-copulas, the mass distribution is a popular object of investigation [4,6]. In this respect, the extremes of this mass distribution are of particular interest. In case $n=2$, the main result is that there exists a unique 2-box on which the minimal mass (which turns out to be $-1/3$) can be accumulated, as well as a unique 2-box (the unit square itself) on which the maximal mass 1 can be accumulated. In order to gain further insight into the difference between n-copulas and n-quasi-copulas, the case $n=3$ seems to be a crucial step, as it involves an interplay between the bivariate and the trivariate case (see e.g. the compatibility problem in [5]). Unfortunately, the extremes of the mass distribution of a 3-quasi-copula have not yet been identified, mainly due to the seemingly complex nature of this optimization problem. The main message of this contribution is that by formulating this optimization problem as a linear programming problem, we can rely on powerful linear optimization tools to tackle it. It is to some extent unexpected that as in the bivariate case, there still exists a unique 3-box on which the minimal mass (which now turns out to be $-4/5$) can be accumulated, while there exist multiple 3-boxes on which the maximal mass 1 can be accumulated. In principle, the methodology exposed here can be applied to the case $n>3$ as well. Apart from an increasing complexity of the formulation, however, no further surprises are to be expected.

2. The mass distribution associated with a 3-quasi-copula

It is known that for any 2-quasi-copula Q and any 2-box B in $[0,1]^2$, it holds that $-1/3\leqslant V_Q(B)\leqslant 1$. Furthermore, $V_Q(B)=1$ if, and only if, $B=[0,1]^2$; and $V_Q(B)=-1/3$ implies that $B=[1/3,2/3]^2$ [6]. We now aim at finding the corresponding results for 3-quasi-copulas.

2.1. Minimal mass

Theorem 2.1. *Consider a* 3*-quasi-copula* Q *and a* 3*-box* B *in* $[0,1]^3$*. It holds that* $-4/5\leqslant V_Q(B)$*, and* $V_Q(B)=-4/5$ *implies that* $B=[2/5,4/5]^3$.

Proof. Let $B=[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]$. We define the length of the edges of B as $\lambda=b_1-a_1$, $\mu=b_2-a_2$, $\nu=b_3-a_3$, and write the value of Q at the vertices of B as $s=Q(a_1,a_2,a_3)$, $s+\alpha=Q(b_1,a_2,a_3)$, $s+\beta=Q(a_1,b_2,a_3)$, $s+\gamma=Q(a_1,a_2,b_3)$, $s+\delta=Q(b_1,b_2,a_3)$, $s+\epsilon=Q(b_1,a_2,b_3)$, $s+\eta=Q(a_1,b_2,b_3)$ and $s+\zeta=Q(b_1,b_2,b_3)$. We choose $a_1,a_2,a_3,\lambda,\mu,\nu,s,\alpha,\beta,\gamma,\delta,\epsilon,\eta,\zeta$ as fundamental parameters. All of them, as well as $a_1+\lambda$, $a_2+\mu$ and $a_3+\nu$ must lie in $[0,1]$. Since Q is increasing and 1-Lipschitz, it must hold that

$$0\leqslant\alpha\leqslant\lambda\leqslant 1-a_1\leqslant 1,\quad 0\leqslant\beta\leqslant\mu\leqslant 1-a_2\leqslant 1,\quad 0\leqslant\gamma\leqslant\nu\leqslant 1-a_3\leqslant 1, \tag{1}$$

and

$$\begin{aligned}
&\max(\alpha,\beta)\leqslant\delta\leqslant\min(\alpha+\mu,\beta+\lambda),\\
&\max(\alpha,\gamma)\leqslant\epsilon\leqslant\min(\alpha+\nu,\gamma+\lambda),\\
&\max(\beta,\gamma)\leqslant\eta\leqslant\min(\beta+\nu,\gamma+\mu),\\
&\max(\delta,\epsilon,\eta)\leqslant\zeta\leqslant\min(\delta+\nu,\epsilon+\mu,\eta+\lambda).
\end{aligned}\tag{2}$$

Moreover, in each of the eight vertices (x, y, z) of B, Q is bounded from below by $\max(x + y + z - 2, 0)$ and from above by $\min(x, y, z)$, i.e.

$$\begin{aligned}
&a_1 + a_2 + a_3 - 2 \leqslant s \leqslant \min(a_1, a_2, a_3),\\
&a_1 + a_2 + a_3 + \lambda - 2 \leqslant s + \alpha \leqslant \min(a_1 + \lambda, a_2, a_3),\\
&a_1 + a_2 + a_3 + \mu - 2 \leqslant s + \beta \leqslant \min(a_1, a_2 + \mu, a_3),\\
&a_1 + a_2 + a_3 + \nu - 2 \leqslant s + \gamma \leqslant \min(a_1, a_2, a_3 + \nu),\\
&a_1 + a_2 + a_3 + \lambda + \mu - 2 \leqslant s + \delta \leqslant \min(a_1 + \lambda, a_2 + \mu, a_3),\\
&a_1 + a_2 + a_3 + \lambda + \nu - 2 \leqslant s + \epsilon \leqslant \min(a_1 + \lambda, a_2, a_3 + \nu),\\
&a_1 + a_2 + a_3 + \mu + \nu - 2 \leqslant s + \eta \leqslant \min(a_1, a_2 + \mu, a_3 + \nu),\\
&a_1 + a_2 + a_3 + \lambda + \mu + \nu - 2 \leqslant s + \zeta \leqslant \min(a_1 + \lambda, a_2 + \mu, a_3 + \nu).
\end{aligned} \tag{3}$$

Finally, the mass accumulated by Q on B is given by

$$V_Q(B) = \zeta - (\delta + \epsilon + \eta) + (\alpha + \beta + \gamma). \tag{4}$$

We now look for those 3-boxes B that minimize $V_Q(B)$. Regarding (4) as a linear objective function of the fundamental parameters and the conditions (1), (2) and (3) as linear inequality constraints, the problem of minimizing $V_Q(B)$ is a linear programming problem. It can be solved, for instance, by means of the simplex method which is implemented in most optimization software packages [2]. The problem of minimizing $V_Q(B)$ subject to the constraints (1)–(3) has the unique solution announced. □

As far as the box on which minimal (negative) mass can be accumulated, the symmetry observed in the bivariate case is no longer present in the trivariate case. Note that the 3-box $[2/5, 4/5]^3$ is still symmetrical w.r.t. the main diagonal. We now provide an example of a 3-quasi-copula Q for which it effectively holds that $V_Q([2/5, 4/5]^3) = -4/5$.

Example 1. Since for the unique solution of Theorem 2.1 it holds that $s = \alpha = \beta = \gamma = 0$ and $\delta = \epsilon = \eta = \zeta = 2/5$, we can define a suitable Q by distributing mass uniformly on $[0, 1]^3$ in the following manner: $2/5$ of (positive) mass on the 3-boxes $[2/5, 4/5]^2 \times [0, 2/5]$, $[0, 2/5] \times [2/5, 4/5]^2$ and $[2/5, 4/5] \times [0, 2/5] \times [2/5, 4/5]$; $1/5$ of (positive) mass on the 3-boxes $[2/5, 4/5] \times [4/5, 1] \times [2/5, 4/5]$, $[4/5, 1] \times [2/5, 4/5]^2$ and $[2/5, 4/5]^2 \times [4/5, 1]$; $-4/5$ of (negative) mass on the 3-box $[2/5, 4/5]^3$; and 0 on the remaining 3-boxes.

2.2. Maximal mass

By maximizing the objective function $V_Q(B)$, or, equivalently, minimizing $-V_Q(B)$, subject to the same boundary conditions (1)–(3) as before, we are able, by means of the simplex method, for instance, to characterize all 3-boxes on which maximal mass can be accumulated. This linear programming problem has infinitely many solutions with maximal mass 1. In contrast to the bivariate case there is no longer a unique 3-box on which the total mass can be accumulated.

Theorem 2.2. *Consider a* 3*-quasi-copula Q and a* 3*-box B in* $[0, 1]^3$. *It holds that* $V_Q(B) \leqslant 1$ *and* $V_Q(B) = 1$ *implies that* $B = [a, 1]^3$, *with* $a \in [0, 1/2]$.

Example 2. We can define a 3-quasi-copula Q by distributing mass uniformly on $[0, 1]^3$ in the following manner: 1 of (positive) mass on the 3-box $[1/2, 1]^3$; $1/2$ of (positive) mass on the 3-boxes $[0, 1/2]^2 \times [1/2, 1]$, $[0, 1/2] \times [1/2, 1] \times [0, 1/2]$ and $[1/2, 1] \times [0, 1/2]^2$; 0 on the 3-box $[0, 1/2]^3$; and $-1/2$ of (negative) mass on each of the remaining 3-boxes.

2.3. Boxes that have one face or edge on the boundary of the unit cube

In the bivariate case, the mass accumulated on 2-boxes that have one edge on the boundary of the unit square is always positive. We therefore wonder about the minimal mass that can be accumulated on 3-boxes that have one face on the boundary of the unit cube – i.e. at least one of a_1, a_2, a_3 is 0, or one of b_1, b_2, b_3 is 1 – or that have one edge on the boundary of the unit cube – i.e. at least two of a_1, a_2, a_3 are 0, or two of b_1, b_2, b_3 are 1. In each case, the problem of minimizing $V_Q(B)$ can be formulated as a linear programming problem and can be solved by standard methods. The cases not mentioned explicitly in the following theorem can be obtained by invoking symmetry arguments:

Theorem 2.3. *Consider a* 3*-quasi-copula* Q*. If* B *is a* 3*-box in* $[0,1]^3$ *of the type*

(i) $B_0 = [0, b_1] \times [a_2, b_2] \times [a_3, b_3]$*, then* $V_Q(B_0) \geqslant -1/2$ *and* $V_Q(B_0) = -1/2$ *implies that* $B_0 = [0, 1/2] \times [1/2, 1] \times [1/2, 1]$;

(ii) $B_{00} = [0, b_1] \times [0, b_2] \times [a_3, b_3]$*, then* $V_Q(B_{00}) \geqslant 0$ *and* $V_Q(B_{00}) = 0$ *implies that* $B_{00} = [0, b_1] \times [0, b_2] \times [a_3, b_3]$ *with* $b_1 + b_2 + b_3 - a_3 \leqslant 2$;

(iii) $B_1 = [a_1, 1] \times [a_2, b_2] \times [a_3, b_3]$*, then* $V_Q(B_1) \geqslant -2/3$ *and* $V_Q(B_1) = -2/3$ *implies that* $B_1 = [a, 1] \times [1/3, 2/3] \times [1/3, 2/3]$ *with* $a \in [1/3, 2/3]$;

(iv) $B_{11} = [a_1, 1] \times [a_2, 1] \times [a_3, b_3]$*, then* $V_Q(B_{11}) \geqslant -1/2$ *and* $V_Q(B_{11}) = -1/2$ *implies that* $B_{11} = [1/2, 1] \times [1/2, 1] \times [a, a + 1/2]$ *with* $a \in [0, 1/2]$.

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