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Algebra

Some matrix completion problems

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Abstract

Let F be a field and let n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$. Let

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix} \in F^{n\times n},$$

where the blocks $A_{i,j}$ are of type $p_i \times p_j$, $i, j \in \{1, 2, 3\}$, and the blocks in the positions (i, i) are square. We describe the characteristic polynomial of A, when:

(i) $A_{1,1}$, $A_{1,2}$ and $A_{3,3}$ are fixed and the other blocks vary;

(ii) $A_{1,1}$, $A_{1,2}$ and $A_{2,3}$ are fixed and the remaining blocks vary. ***To cite this article: G. Cravo, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Quelques problèmes de complétion de matrices. Soit F un corps et soient n, p_1, p_2, p_3 des entiers positifs tels que $n = p_1 + p_2 + p_3$. Soit

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix} \in F^{n\times n},$$

où les blocs $A_{i,j}$ sont de type $p_i \times p_j$, $i, j \in \{1, 2, 3\}$. Nous étudions le polynôme caractéristique de la matrice A, quand :

(i) $A_{1,1}$, $A_{1,2}$ et $A_{3,3}$ sont connus et les autres blocs varient ;

(ii) $A_{1,1}$, $A_{1,2}$ et $A_{2,3}$ sont connus et les autres blocs varient. ***Pour citer cet article : G. Cravo, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

1. Motivation

In the theory of matrices, there exists an important class of problems called the Matrix Completion Problems. These problems arose from a large variety of applications, such as control theory, control design, geophysics, circuit

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theory. As an example the Pole Assignment Problem is a very important problem in control theory. The general idea of this type of problems consists in studying the possibility to 'complete' a matrix, when some of its entries are prescribed (i.e., are fixed), such that the resulting matrix satisfies certain properties. In this context 'to complete' means to attribute values to the remaining entries. Throughout the last decades these type of problems have often been studied by many authors.

One important problem in Matrix Completion Problems, consists to studying the spectrum of an $n \times n$ matrix, with entries in a field F, when some of its entries are fixed and the others vary. One of the earliest results concerning this subject was obtained by Schur in 1923. The author proved that the diagonal of an $n \times n$ Hermitian matrix with spectrum $S = \{c_1, \ldots, c_n\}$ is contained in the convex hull of symmetric group actions on S. In 1954, A. Horn, proved that each point in the convex hull can be obtained in the same way, that is, as the diagonal of some Hermitian matrix. Later, other authors, such as L. Mirsky, H.K. Farahat and W. Lederman, D. London and H. Minc, G.N. Oliveira studied this problem. The most general result concerning this question was obtained by D. Hershkowitz in [4]. The author showed that it is possible to prescribe $2n - 3$ entries of the matrix, simultaneously with the spectrum of the matrix, except in some exceptional cases.

Another important problem, that motivates our work is the following, a particular case of the Matrix Completion Problems, proposed by G.N. Oliveira in 1975.

Let F be a field. We denote by $F^{p\times q}$ the set of all matrices of type $p \times q$, with entries in F.

Problem. [5] Let n, p, q be positive integers such that $n = p+q$. Let $f(x) \in F[x]$ be a monic polynomial of degree n. Let

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \in F^{n\times n}, \tag{1}$$

be a partitioned matrix, where $A_{1,1} \in F^{p\times p}$, $A_{2,2} \in F^{q\times q}$. Suppose that some of the blocks $A_{i,j}$, $i, j \in \{1, 2\}$, are known. Under which conditions does there exist a matrix of the form (1) with characteristic polynomial $f(x)$?

Note that this problem can be split into essentially seven distinct problems, according to the prescription of some blocks of A: $A_{1,1}$ prescribed; $A_{1,2}$ prescribed; $A_{1,1}$ and $A_{1,2}$ prescribed; $A_{1,1}$ and $A_{2,2}$ prescribed; $A_{1,2}$ and $A_{2,1}$ prescribed; $A_{1,1}$, $A_{1,2}$ and $A_{2,2}$ prescribed; $A_{1,1}$, $A_{1,2}$ and $A_{2,1}$ prescribed. Some of these problems are completely solved, but there are cases for which there exist only partial solutions.

Note that if all the roots of $f(x)$ are in F, these problems consist in studying the eigenvalues of a matrix of the form (1) when some of its blocks are fixed and the remaining blocks vary.

Motivated by the fascination of this type of problems, we had the purpose to solve the following problem.

Let F be an arbitrary field. Let $n, k, p_1, \ldots, p_k$ be positive integers such that $n = p_1 + \cdots + p_k$. Let $c_1, \ldots, c_n \in F$. Let

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,k} \\ \vdots & & \vdots \\ A_{k,1} & \cdots & A_{k,k} \end{bmatrix} \in F^{n\times n}, \tag{2}$$

where the blocks $A_{i,j} \in F^{p_i\times p_j}$, $i, j \in \{1, \ldots, k\}$, and $A_{1,1}, \ldots, A_{k,k}$ are square submatrices.

Problem. Suppose that some of the blocks $A_{i,j}$ are prescribed. Under which conditions does there exist a matrix of the form (2) with eigenvalues $c_1, \ldots, c_n$?

This problem is a unification of the previous problems and was studied by the present author in her PhD thesis [3], by supervision of F.C. Silva (also see the references). The study of this problem when arbitrary blocks are prescribed, becomes very difficult. In fact if the prescribed positions correspond to 'large' submatrices, then there are necessary interlacing inequalities involving invariant factors [6,7]. In this paper we present some results obtained for the case $k = 3$.

2. Main results

From now on, suppose that $k = 3$. Let F be an algebraically closed field and let $f \in F[x]$ be a monic polynomial of degree n.

In the following result we describe the characteristic polynomial of a matrix of the form (2), when $A_{1,1}, A_{1,2}, A_{3,3}$ are fixed and the remaining blocks vary.

Theorem 2.1. [1] *Let $A_{1,1} \in F^{p_1 \times p_1}$, $A_{1,2} \in F^{p_1 \times p_2}$, $A_{3,3} \in F^{p_3 \times p_3}$. Let $\alpha_1 \mid \cdots \mid \alpha_{p_1}$ be the invariant factors of*

$$[xI_{p_1} - A_{1,1} | - A_{1,2}] \tag{3}$$

and let $\beta_1 \mid \cdots \mid \beta_{p_3}$ be the invariant polynomials of $A_{3,3}$.

If $p_1 \leqslant p_3 \leqslant p_1 + p_2$, there always exist $A_{1,3} \in F^{p_1 \times p_3}$, $A_{2,1} \in F^{p_2 \times p_1}$, $A_{2,2} \in F^{p_2 \times p_2}$, $A_{2,3} \in F^{p_2 \times p_3}$, $A_{3,1} \in F^{p_3 \times p_1}$, $A_{3,2} \in F^{p_3 \times p_2}$ such that the matrix of the form (2) *has characteristic polynomial f.*

Now consider the following conditions:

($\mathrm{i}_{2.1}$) *If $p_1 > p_3$, then $\alpha_1 \cdots \alpha_{p_1 - p_3} | f$;*
($\mathrm{ii}_{2.1}$) *If $p_3 > p_1 + p_2$, then $\beta_1 \cdots \beta_{p_3 - p_1 - p_2} | f$.*

Then there exists $A_{1,3} \in F^{p_1 \times p_3}$, $A_{2,1} \in F^{p_2 \times p_1}$, $A_{2,2} \in F^{p_2 \times p_2}$, $A_{2,3} \in F^{p_2 \times p_3}$, $A_{3,1} \in F^{p_3 \times p_1}$, $A_{3,2} \in F^{p_3 \times p_2}$ such that the matrix of the form (2) *has characteristic polynomial f if and only if either* ($\mathrm{i}_{2.1}$) *or* ($\mathrm{ii}_{2.1}$) *is satisfied.*

The next two lemmas were established to simplify the proof of Theorem 2.4, where we describe the characteristic polynomial of a matrix of the form (2), when $A_{1,1}, A_{1,2}, A_{2,3}$ are fixed and the other blocks vary.

Lemma 2.2. [2] *Let $A_{1,1} \in F^{p_1 \times p_1}$, $A_{1,2} \in F^{p_1 \times p_2}$ and let $\alpha_1 | \cdots | \alpha_{p_1}$ be the invariant factors of* (3). *Let t be the greatest index in $\{0, 1, \ldots, p_1\}$ such that $\alpha_t = 1$. Let $r = \operatorname{rank} A_{1,2}$ and let $k_1 \geqslant \cdots \geqslant k_r > k_{r+1} = \cdots = k_{p_2} (= 0)$ be the minimal indices for the columns of* (3). *Let $s = k_1 + \cdots + k_{p_2}$. Then, there exists a nonsingular matrix $P \in F^{p_1 \times p_1}$ such that*

$$P^{-1} \begin{bmatrix} A_{1,1} & A_{1,2} \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & I_{p_2} \end{bmatrix} = \left[\begin{array}{cc|c} N & 0 & 0 \\ T & M_1 & M_2 \end{array}\right],$$

where $N = C(\alpha_{t+1}) \oplus \cdots \oplus C(\alpha_{p_1}) \in F^{(p_1 - s) \times (p_1 - s)}$, (M_1, M_2) is a completely controllable pair, where $M_1 \in F^{s \times s}$, $M_2 \in F^{s \times p_2}$ and $T \in F^{s \times (p_1 - s)}$.

Lemma 2.3. [2] *Let $A_{1,1} \in F^{p_1 \times p_1}$, $A_{1,2} \in F^{p_1 \times p_2}$, $A_{2,3} \in F^{p_2 \times p_3}$. Let $P \in F^{p_1 \times p_1}$ be a nonsingular matrix such that*

$$\begin{bmatrix} A'_{1,1} & A'_{1,2} \end{bmatrix} = P^{-1} \begin{bmatrix} A_{1,1} & A_{1,2} \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & I_{p_2} \end{bmatrix}.$$

Then there exists $A_{1,3} \in F^{p_1 \times p_3}$, $A_{2,1} \in F^{p_2 \times p_1}$, $A_{2,2} \in F^{p_2 \times p_2}$, $A_{3,1} \in F^{p_3 \times p_1}$, $A_{3,2} \in F^{p_3 \times p_2}$, $A_{3,3} \in F^{p_3 \times p_3}$ such that the matrix of the form (2) *has characteristic polynomial f if and only if there exists $A'_{1,3} \in F^{p_1 \times p_3}$, $A'_{2,1} \in F^{p_2 \times p_1}$, $A'_{2,2} \in F^{p_2 \times p_2}$, $A'_{3,1} \in F^{p_3 \times p_1}$, $A'_{3,2} \in F^{p_3 \times p_2}$, $A'_{3,3} \in F^{p_3 \times p_3}$ such that*

$$A' = \begin{bmatrix} A'_{1,1} & A'_{1,2} & A'_{1,3} \\ A'_{2,1} & A'_{2,2} & A_{2,3} \\ A'_{3,1} & A'_{3,2} & A'_{3,3} \end{bmatrix}$$

has characteristic polynomial f.

Theorem 2.4. [2] *Let $A_{1,1} \in F^{p_1 \times p_1}$, $A_{1,2} \in F^{p_1 \times p_2}$, $A_{2,3} \in F^{p_2 \times p_3}$. Let $\alpha_1 | \cdots | \alpha_{p_1}$ be the invariant factors of* (3). *Then there exists $A_{1,3} \in F^{p_1 \times p_3}$, $A_{2,1} \in F^{p_2 \times p_1}$, $A_{2,2} \in F^{p_2 \times p_2}$, $A_{3,1} \in F^{p_3 \times p_1}$, $A_{3,2} \in F^{p_3 \times p_2}$, $A_{3,3} \in F^{p_3 \times p_3}$ such that the matrix of the form* (2) *has characteristic polynomial f if and only if*

$$\alpha_1 \cdots \alpha_{p_1 - p_3} | f.$$

(*Make convention that* $\alpha_1 \cdots \alpha_{p_1-p_3} = 1$ *if* $p_1 \leqslant p_3$.)

Remark 1. According to the previous lemmas, to prove Theorem 2.4, we may assume, without loss of generality, that $[\, A_{1,1} \quad A_{1,2} \,]$ has the form

$$\left[\begin{array}{cc|c} N & 0 & 0 \\ T & M_1 & M_2 \end{array}\right],$$

where N, M_1, M_2 and T satisfy the conditions of Lemma 2.2.

Remark 2. Theorems 2.1 and 2.4 are extensions of Oliveira's problem [5], when F is an algebraically closed field.

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