Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 344 (2007) 749–752
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Complex Analysis

Boundedness of Hankel operators on $\mathcal{H}^1(\mathbb{B}^n)$

Aline Bonami [a], Sandrine Grellier [a], Benoît F. Sehba [b]

[a] *Fédération Denis-Poisson, MAPMO-UMR 6628, département de mathématiques, université d'Orléans, 45067 Orléans cedex 2, France*
[b] *Department of Mathematics, University of Glasgow, Glasgow G12 8QW, UK*

Received 9 February 2007; accepted after revision 15 May 2007

Available online 22 June 2007

Presented by Yves Meyer

Abstract

We prove that the Hankel operator h_b associated to the Szegö projection on the unit ball $\mathbb{B}^n$ is bounded on the Hardy space $\mathcal{H}^1(\mathbb{B}^n)$ if and only if its symbol b has logarithmic mean oscillation on the unit sphere. ***To cite this article: A. Bonami et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Continuité de l'opérateur de Hankel sur $\mathcal{H}^1(\mathbb{B}^n)$. On démontre que l'opérateur de Hankel h_b associé au projecteur de Szegö sur la boule unité s'étend continûment à l'espace de Hardy $\mathcal{H}^1(\mathbb{B}^n)$ si et seulement si b est à oscillation moyenne logarithmique sur la sphère unité. ***Pour citer cet article : A. Bonami et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

1. Introduction

Let b be a holomorphic function in $\mathcal{H}^2(\mathbb{B}^n)$. The little Hankel operator with symbol b is defined for f a bounded holomorphic function by

$$h_b(f) := P(b\bar{f}).$$

Here P denotes the Szegö projection.

We prove that h_b extends to a bounded operator on $\mathcal{H}^1(\mathbb{B}^n)$ if and only if $b \in \mathrm{LMOA}$. This result generalizes the corresponding result in the unit disc, proved in [3] and [6]. The key of the proof of the necessary condition is a factorization of functions, which is obtained by a modification of the one of [2]. This weak factorization is developed and generalized in a forthcoming paper for all strictly pseudo-convex domains or convex domains of finite type in $\mathbb{C}^n$ [1]. Our aim, here, is to give a self-contained simple proof for the unit ball, which relies on the notion of logarithmic Carleson measure for the sufficient condition.

E-mail addresses: Aline.Bonami@univ-orleans.fr (A. Bonami), Sandrine.Grellier@univ-orleans.fr (S. Grellier), bs@maths.gla.ac.uk (B.F. Sehba).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2007.05.004

Let us recall some basic facts and notations. We denote by $B_\delta(\xi)$ the anisotropic ball on the unit sphere $\mathbb{S}^n$, that is, the set

$$B_\delta(\xi) = \{\zeta \in \mathbb{S}^n;\ |1 - \langle \xi, \zeta\rangle| \leqslant \delta\}$$

and by $Q_\delta(\xi)$ the 'tent' over the ball $B_\delta(\xi)$, that is, the set

$$Q_\delta(\xi) = \{z \in \mathbb{B}^n;\ |1 - \langle \xi, z\rangle| \leqslant \delta\}.$$

We denote by $\mathrm{d}V$ the Lebesgue measure in $\mathbb{B}^n$, and by $\mathrm{d}\sigma$ the normalized Euclidean measure on $\mathbb{S}^n$. For $f \in \mathcal{H}^1(\mathbb{B}^n)$, we also note $f(\xi)$, for $\xi \in \mathbb{S}^n$, the admissible limit at the boundary, which exists a.e. Recall that LMOA is the space of functions $f \in \mathcal{H}^1(\mathbb{B}^n)$ with logarithmic mean oscillation at the boundary. More precisely, for $f \in$ LMOA, there exists a constant $C > 0$ so that, for any ball $B = B_\delta(\xi)$ with $\xi \in \mathbb{S}_n$ and $0 < \delta < 1$,

$$\frac{1}{\sigma(B)} \int_B |f - f_B| \,\mathrm{d}\sigma \leqslant \frac{C}{\log(4/\delta)}.$$

Here f_B denotes the mean-value of f on B. It is well known that f belongs to LMOA if and only if $\mathrm{d}\mu(z) = (1 - |z|^2)|\nabla f(z)|^2 \,\mathrm{d}V(z)$ is a logarithmic Carleson measure (see [5]), that is, there exists some constant C such that, for all $\xi \in \mathbb{S}_n$ and $0 < \delta < 1$,

$$\mu(Q_\delta(\xi)) \leqslant C \frac{\sigma(B_\delta(\xi))}{(\log(4/\delta))^2}.$$

Theorem 1.1. *The Hankel operator h_b extends into a bounded operator on $\mathcal{H}^1(\mathbb{B}^n)$ if and only if $b \in$ LMOA.*

2. The necessary condition

Let us first assume that h_b is bounded on $\mathcal{H}^1(\mathbb{B}^n)$ and prove that there exists a constant $C > 0$ so that, for any ball B of radius δ on $\mathbb{S}^n$,

$$\frac{1}{\sigma(B)} \int_B |b - b_B| \,\mathrm{d}\sigma \leqslant \frac{C}{\log(4/\delta)} \|h_b\|_{\mathcal{H}^1(\mathbb{B}^n) \to \mathcal{H}^1(\mathbb{B}^n)}.$$

It is sufficient to show that, for any bounded function a supported in B with $\|a\|_\infty \leqslant \sigma(B)^{-1} \log(4/\delta)$,

$$\left| \int_B (b - b_B)\bar{a} \,\mathrm{d}\sigma \right| \leqslant C \|h_b\|_{\mathcal{H}^1(\mathbb{B}^n) \to \mathcal{H}^1(\mathbb{B}^n)}.$$

Without loss of generality we may assume that a has mean-value zero and, since b is holomorphic, replace a by its projection Pa in the left-hand side. Finally, we want to prove that

$$\left| \int_{\mathbb{S}^n} b\overline{Pa} \,\mathrm{d}\sigma \right| \leqslant C \|h_b\|_{\mathcal{H}^1(\mathbb{B}^n) \to \mathcal{H}^1(\mathbb{B}^n)}.$$

If we can write $Pa = u \times v$ with $u \in$ BMOA and $v \in \mathcal{H}^1(\mathbb{B}^n)$ such that $\|u\|_{\mathrm{BMOA}} \leqslant C$ and $\|v\|_{\mathcal{H}^1} \leqslant C$, we conclude easily by writing

$$\left| \int_{\mathbb{S}^n} b\overline{Pa} \,\mathrm{d}\sigma \right| = |\langle b, uv\rangle| = |\langle b\bar{v}, u\rangle| = |\langle h_b(v), u\rangle| \leqslant C^2 \|h_b\|_{\mathcal{H}^1(\mathbb{B}^n) \to \mathcal{H}^1(\mathbb{B}^n)},$$

using the duality $(\mathcal{H}^1(\mathbb{B}^n)$, BMOA). So, let us find u and v. We assume that B is centered at $\xi_0 \in \mathbb{S}^n$. We put $u(z) := \log(1 - \langle z, (1-\delta)\xi_0\rangle)$. It is well known that this function belongs uniformly to BMOA or, equivalently, that $(1 - |z|^2)|\nabla u(z)|^2 \,\mathrm{d}V(z)$ is uniformly a Carleson measure.

It remains to prove that $v := (Pa)/u$ belongs to $\mathcal{H}^1(\mathbb{B}^n)$. We need to estimate $\int_{\mathbb{S}^n} |v(z)| \,\mathrm{d}\sigma(z)$, which we split into the integral on $\widetilde{B}$, where $\widetilde{B}$ is the ball $B_{2\delta}(\xi_0)$, and the integral on the complement. For $z \in \widetilde{B}$, we have $|1 - \langle z, (1 -$

$\delta)\xi_0\rangle| \simeq \delta$, and $|u(z)| \simeq \log(4/\delta)$. It classically follows from Schwarz inequality and from the fact that P is bounded in L^2 that

$$\int_{\widetilde{B}} |v|\,\mathrm{d}\sigma(z) \leqslant \frac{C}{\log(4/\delta)}\big(\sigma(B)\big)^{1/2}\|Pa\|_{L^2(\mathbb{S}^n)} \leqslant C.$$

This concludes for the first term. For the second one, we use the following estimates for z outside $\widetilde{B}$ (see [4] for Pa)

$$\big|Pa(z)\big| \leqslant C\delta^{1/2}\|a\|_\infty\sigma(B)\big|1-\langle z,\xi_0\rangle\big|^{-n-1/2},$$

while $|u(z)| \simeq \log(4/|1-\langle z,\xi_0\rangle|)$. To conclude for the boundedness of $\int_{{}^c\widetilde{B}} |v(z)|\,\mathrm{d}\sigma(z)$, we use the fact that

$$\delta^{1/2}\log(4/\delta)\int_{{}^c\widetilde{B}} \big|1-\langle z,\xi_0\rangle\big|^{-n-1/2}\big(\log\big(4/\big|1-\langle z,\xi_0\rangle\big|\big)\big)^{-1}\,\mathrm{d}\sigma(z) \leqslant C.$$

This finishes the proof of the fact that b belongs to LMOA.

3. The sufficient condition

We first give an equivalent definition of logarithmic Carleson measures (see [7]):

Proposition 1. *Let μ be a positive Borel measure on $\mathbb{B}^n$. Then the following conditions are equivalent.*

(i) *The measure μ is a logarithmic Carleson measure.*
(ii) *There is $C > 0$ such that for any $g \in$ BMOA and any $f \in \mathcal{H}^2(\mathbb{B}^n)$,*

$$\int_{\mathbb{B}^n} \big|g(z)\big|^2\big|f(z)\big|^2\,\mathrm{d}\mu(z) \leqslant C\|g\|^2_{\mathrm{BMOA}}\|f\|^2_{\mathcal{H}^2}.$$

We need also the following lemma, which follows easily from integration by parts (see also [4]). Here R denotes the radial derivative:

Lemma 2. *Let φ, ψ be holomorphic polynomials on $\mathbb{B}^n$. Then the following equality holds*

$$\int_{\mathbb{S}^n} \varphi(\xi)\overline{\psi(\xi)}\,\mathrm{d}\sigma(\xi)$$
$$= c_0\int_{\mathbb{B}^n} \varphi(z)\overline{\psi(z)}\,\mathrm{d}V(z) + c_1\int_{\mathbb{B}^n} R\varphi(z)\overline{\psi(z)}\big(1-|z|^2\big)\,\mathrm{d}V(z) + c_2\int_{\mathbb{B}^n} R\varphi(z)\overline{R\psi(z)}\big(1-|z|^2\big)\,\mathrm{d}V(z).$$

Proof of the sufficiency of the condition. Let b in LMOA. For $f \in \mathcal{H}^1(\mathbb{B}^n)$ and $g \in$ BMOA, we want to estimate $|\langle h_b(f), g\rangle| = |\langle b, fg\rangle|$. We use Lemma 2 for functions $\varphi = b$ and $\psi = fg$ (which we may assume smooth enough so that the identity is valid). So we have to estimate the three terms of the integral. For the first one, since b and g are in all Hardy spaces $\mathcal{H}^p(\mathbb{B}^n)$, the product $|b(z)g(z)|$ is bounded by $C(1-|z|^2)^{-1/2}$, and we conclude directly. For the two other terms, we have to consider

$$I_1 := \int_{\mathbb{B}^n} \big|f(z)\big|\big(\big|g(z)\big| + \big|\nabla g(z)\big|\big)\big|\nabla b(z)\big|\big(1-|z|^2\big)\,\mathrm{d}V(z),$$

$$I_2 := \int_{\mathbb{B}^n} \big|g(z)\big|\big|\nabla f(z)\big|\big|\nabla b(z)\big|\big(1-|z|^2\big)\,\mathrm{d}V(z).$$

For I_1, we use Schwarz inequality to obtain

$$I_1^2 \leqslant C\int_{\mathbb{B}^n} \big|f(z)\big|\big(\big|g(z)\big|^2 + \big|\nabla g(z)\big|^2\big)\big(1-|z|^2\big)\,\mathrm{d}V(z) \times \int_{\mathbb{B}^n} \big|f(z)\big|\big|\nabla b(z)\big|^2\big(1-|z|^2\big)\,\mathrm{d}V(z).$$

We conclude by using the fact that $(1-|z|^2)|\nabla b(z)|^2\,\mathrm{d}V(z)$, $(1-|z|^2)|\nabla g(z)|^2\,\mathrm{d}V(z)$ and $|g(z)|^2(1-|z|^2)\,\mathrm{d}V(z)$ are Carleson measures.

The main point is to estimate I_2. We first recall that, by the weak factorization theorem (see [4]), any $f \in \mathcal{H}^1(\mathbb{B}^n)$ can be written as

$$f = \sum_j h_j l_j \quad \text{with} \sum_j \|h_j\|_{\mathcal{H}^2}\|l_j\|_{\mathcal{H}^2} \leqslant C\|f\|_{\mathcal{H}^1}.$$

Replacing f by this weak factorization, we are led to estimate a sum of terms like

$$J := \int_{\mathbb{B}^n} \big|g(z)\big|\big|l(z)\big|\big|\nabla h(z)\big|\big|\nabla b(z)\big|\big(1-|z|^2\big)\,\mathrm{d}V(z)$$

for l and h in $\mathcal{H}^2(\mathbb{B}^n)$. We recall that, for $h \in \mathcal{H}^2(\mathbb{B}^n)$,

$$\int_{\mathbb{B}^n} \big|\nabla h(z)\big|^2\big(1-|z|^2\big)\,\mathrm{d}V(z) \leqslant C\|h\|^2_{\mathcal{H}^2}.$$

Using this last inequality, Schwarz inequality and Proposition 1, we obtain

$$J \leqslant C\|h\|_{\mathcal{H}^2}\left(\int_{\mathbb{B}^n} \big|g(z)\big|^2\big|l(z)\big|^2\big|\nabla b(z)\big|^2\big(1-|z|^2\big)\,\mathrm{d}V(z)\right)^{1/2} \leqslant C\|g\|_{\mathrm{BMOA}}\|l\|_{\mathcal{H}^2}\|h\|_{\mathcal{H}^2}.$$

This allows us to conclude. □

References

[1] A. Bonami, S. Grellier, Decomposition theorems for Hardy–Orlicz spaces and weak factorization, Preprint, 2007.
[2] A. Bonami, M. Peloso, F. Symesak, Factorization of Hardy spaces and Hankel operators on convex domains in $\mathbb{C}^n$, J. Geom. Anal. 11 (3) (2001) 363–397.
[3] S. Janson, J. Peetre, S. Semmes, On the action of Hankel and Toeplitz operators on some function spaces, Duke Math. J. 51 (4) (1984) 937–958.
[4] W. Rudin, Function Theory in the Unit Ball of C^n, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Science), vol. 241, Springer-Verlag, New York–Berlin, ISBN: 0-387-90514-6, 1980, xiii+436 pp.
[5] W.S. Smith, BMO(ρ) and Carleson measures, Trans. Amer. Math. Soc. 287 (1) (1985) 107–126.
[6] V.A. Tolokonnikov, Hankel and Toeplitz operators in Hardy spaces, Soviet Math. 37 (1987) 1359–1364.
[7] R. Zhao, On logarithmic Carleson measures, Acta Sci. Math. (Szeged) 69 (2003) 605–618.