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Partial Differential Equations

The Webster scalar curvature problem on higher dimensional CR compact manifolds

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Abstract

Using a topological arguments due to Aubin–Bahri (1997), we give some existence results for the Webster scalar curvature problem on the $2n+1$ dimensional *CR* compact manifolds locally conformally *CR* equivalent to the unit sphere S^{2n+1} of $\mathbb{C}^{n+1}$. ***To cite this article: H. Chtioui, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Le problème de la courbure scalaire de Webster sur les variétés CR. Par des arguments topologiques introduits par Aubin–Bahri (1997), nous donnons quelques résultats d'existence pour le problème de la courbure scalaire de Webster sur les variétés *CR* de dimension $2n+1$ localement conformément *CR* equivalent à la sphère unité S^{2n+1} de $\mathbb{C}^{n+1}$. ***Pour citer cet article : H. Chtioui, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Webster scalar curvature problem

Let (M,θ) be a *CR* compact manifold of dimension $2n+1$ with a contact form θ and let $f : M \to \mathbb{R}$ be a C^3 positive function. The prescribed Webster scalar curvature on M is to find suitable conditions on f such that f is the Webster scalar curvature for some contact form $\tilde{\theta}$ on M conformally equivalent to θ. If we set $\tilde{\theta} = u^{2/n}\theta$, where u is a smooth positive function on M, then the above problem is equivalent to solve the following equation

$$(\mathrm{P})\quad \begin{cases} (2+\frac{2}{n})\Delta_\theta u + R_\theta u = f u^{1+2/n}, \\ u > 0 \quad \text{in } M, \end{cases}$$

where Δ_θ is the sub-Laplacian operator on (M,θ) and R_θ is the Webster scalar curvature of (M,θ).

Few results have been established on problem (P); in [15], Malchiodi and Uguzzoni considered the case where $M = S^{2n+1}$ and gave a perturbative result for problem (P). Their approach uses a perturbation method due to Ambrosetti and Badiale [1]. In [9], Gamara considered the case where M is locally conformally *CR* equivalent to the *CR* sphere of $\mathbb{C}^2$ and provided an Euler–Hoph type criterion for f to find solution of (P) for $n = 1$. The method of [9] is due to

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Bahri and Coron [6]. On the contrary, the Yamabe problem on *CR* manifolds, that is when f is assumed to be constant, has been widely studied see [10–14].

2. New results

In this Note we focus on the higher dimensional *CR* manifolds and we give a contribution in the same direction as in the papers of Aubin–Bahri [2] and Bahri [5] concerning the Riemannian case. Our methods are based on the techniques related to the theory of critical points at infinity (see [4]). We extend these tools to the framework of such Eq. (P).

Through this Note, we assume that f has only nondegenerate critical points $y_0, y_1, \dots, y_\ell$ such that

$$\Delta f(y_i) \neq 0 \quad \text{for } i = 0, \dots, \ell.$$

For $a \in M$ and $\lambda \gg 1$, we define a family of 'almost solutions' $\tilde{\delta}_{(a,\lambda)}$ of the Yamabe problem in M (see section two of [9]).

Our first main result is the following:

Proposition 2.1. *Let $n \geqslant 2$. Assume that* (P) *has no solution. Then the only critical points at infinity of the associated variational problem correspond to*

$$\sum_{j=1}^{p} f(y_{i_j})^{(2-n)/2} \tilde{\delta}_{(y_{i_j},\infty)}$$

with $p \in \mathbb{N}^*$, $y_{i_j} \neq y_{i_k}$ for $j \neq k$ and $-\Delta f(y_{i_j}) > 0$ for $j = 1, \dots, p$.

Notice that Proposition 2.1 should be useful for the study of the existence solutions to problem (P). At this point, we will illustrate its usefulness through the following three results.

Let $F^+ = \{y_i,\ \nabla f(y_i) = 0 \text{ and } -\Delta f(y_i) > 0\}$.

($\mathbf{A_1}$) We assume that

$$f(y_0) \geqslant f(y_1) \geqslant \cdots \geqslant f(y_h) > f(y_{h+1}) \geqslant \cdots \geqslant f(y_\ell),$$

where $F^+ = \{y_0, y_1, \dots, y_h\}$ and $0 \leqslant h \leqslant \ell$.

($\mathbf{A_1'}$) We assume that $y_j \notin F^+$ for all $j \in \{h+1, \dots, \ell\}$. In addition, we assume that for every $i \in \{1, \dots, h\}$, such that $y_i \notin F^+$, we have

$$n - m + 3 \leqslant \operatorname{ind}(f, y_i) \leqslant n - 2,$$

where, $\operatorname{ind}(f, y_i)$ is the Morse index of f at y_i and m is an integer defined in the assumption ($\mathbf{A_2}$).

($\mathbf{A_2}$) We assume that there exists a pseudo-gradient Z for f of Morse–Smale type, (that is the intersection of the stable and the unstable manifolds of the critical points of f are transverse) such that the set X is not contractible, where

$$X = \bigcup_{0 \leqslant i \leqslant h} \overline{W_s(y_i)}$$

and $W_s(y_i)$ is the stable manifold of y_i for Z. We denote by m the dimension of the first nontrivial reduced homology group of X.

($\mathbf{A_3}$) We assume that there exists a positive constant $\bar{c}$ such that $\bar{c} < f(y_h)$ and such that X is deformable to a point in $f^{\bar{c}} = \{x \in M \mid f(x) \geqslant \bar{c}\}$.

We then have:

Theorem 2.1. *Let $n \geqslant 2$. There exists a positive constant c_0 independent of f such that if f satisfies* ($\mathbf{A_1}$)–($\mathbf{A_3}$) *and $f(y_0)/\bar{c} \leqslant 1 + c_0$, then problem* (P) *has a solution.*

Corollary 2.1. *The solution obtained in Theorem* 2.1 *has an augmented Morse index greater than or equal to m.*

Theorem 2.2. *Assume that $n \geqslant 3$. Then, there exists a positive constant c_0 independent of f such that if f satisfies* $(\mathbf{A}_1')$, $(\mathbf{A}_2)$, $(\mathbf{A}_3)$ *and $f(y_0)/\bar{c} \leqslant 1 + c_0$, then problem* (P) *has a solution*

Remark 2.1. The result of Theorem 2.1 is true for $n = 1$, taking F^+ in this case the following set,

$$F^+ = \left\{ y_i \mid \nabla f(y_i) = 0 \text{ and } -\frac{\Delta f(y_i)}{3f(y_i)} - 2Ay_i > 0 \right\}$$

where Ay_i is the value of the regular part of the Green's function of the operator Δ on M evaluated at y_i.

Remark 2.2.

(i) The assumption $n \geqslant 3$ in Theorem 2.2 is needed in order to make $(\mathbf{A}_1')$ meaningful;
(ii) The assumption $f(y_0)/\bar{c} \leqslant 1 + c_0$ allows one, basically, to perform a single-bubble analysis;
(iii) To see how to construct an example of a function H satisfying our assumptions, we refer the reader to [3].

Remark 2.3.

(i) The proof of Proposition 2.1 is quite difficult and extremely technical. In principle, it relies on the construction of a suitable pseudogradient W at infinity as in [5] and [7], which in turn relies on very delicate expansion of the Euler–Lagrange functional associated to (P) and its gradient near infinity.
(ii) The main idea to prove Theorems 2.1 and 2.2 is to compute the topological contribution of the critical points at infinity between the level sets of the associated Euler functional and the main issue is under our conditions on f. There remains some difference of topology not due to the critical points at infinity and therefore the existence of solution to (P). The details of the proof of our results are given in [8].

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