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Topology

# Spin ${ }^{q}$ manifolds and $S^{1}$ actions 

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#### Abstract

We prove a vanishing theorem for $\operatorname{Spin}^{q}$ manifolds admitting $S^{1}$ actions, generalizing those of Atiyah and Hirzebruch for Spin manifolds and Hattori for $\operatorname{Spin}^{c}$ manifolds. We also prove a vanishing theorem for almost quaternionic manifolds with compatible circle actions. To cite this article: H. Herrera, R. Herrera, C. R. Acad. Sci. Paris, Ser. I 345 (2007). © 2007 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.


## Résumé

Variétés $\operatorname{Spin}^{q}$ et actions de $\boldsymbol{S}^{\mathbf{1}}$. On montre un théorème d'annulation pour les variétés $\operatorname{Spin}^{q}$ qui admettent des actions de $S^{1}$, ce qui généralise le théorème d'Atiyah et de Hirzebruch pour les variétés de Spin et celui de Hattori pour les variétés Spin ${ }^{c}$. De plus, on montre un théorème d'annulation pour les variétés presque quaternionienne qui admettent des actions de $S^{1}$ compatibles.
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## 1. Introduction

The orthonormal frame bundle $P_{\text {SO }}$ of a Spin manifold admits a double cover by a bundle $P_{\text {Spin }}$ with fiber a Spin group. The existence of this principal bundle gives rise to the spinor vector bundle and to the Dirac operator, which are intimately linked to the geometry and topology of the manifold. There are manifolds, however, which do not admit Spin structures but admit either $\operatorname{Spin}^{c}$ or $\operatorname{Spin}^{q}$ structures.

In this Note, we prove a vanishing theorem for characteristic numbers and indices of twisted Dirac operators (see Theorem 3.1) generalizing those of Atiyah and Hirzebruch for Spin manifolds [1] and Hattori for Spin ${ }^{c}$ manifolds [4], and a vanishing theorem for almost quaternionic manifolds with compatible $S^{1}$ actions. In Section 2, we recall preliminaries on $\operatorname{Spin}^{q}$ structures. In Section 3 we define twisted Dirac operators on $\operatorname{Spin}^{q}$ manifolds and prove the vanishing Theorem 3.1 for $\operatorname{Spin}^{q}$ manifolds. In Section 4 we prove vanishing Theorem 4.1 for almost quaternionic manifolds.

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## 2. $\mathbf{S p i n}{ }^{q}$ structures

Let $\operatorname{SO}(n)$ denote the special orthogonal group and $\operatorname{Spin}(n)$ its universal double-cover. The Spin group can be 'twisted quaternionically' by using the unit quaternions $\mathrm{Sp}(1)$ as follows

$$
\operatorname{Spin}^{q}(n)=(\operatorname{Spin}(n) \times \operatorname{Sp}(1)) /\{ \pm(1,1)\}=\operatorname{Spin}(n) \times_{\mathbb{Z}_{2}} \operatorname{Sp}(1),
$$

and the short exact sequence $1 \rightarrow \mathbb{Z}_{2} \rightarrow \operatorname{Spin}^{q}(n) \rightarrow \mathrm{SO}(n) \times \mathrm{SO}(3) \rightarrow 1$.
Definition 2.1. Let $M$ be an oriented Riemannian manifold with a fixed metric and let $P_{\mathrm{SO}(n)}(M)$ denote its bundle of oriented orthonormal frames. $M$ is called $\operatorname{Spin}^{q}$ if it admits a $\operatorname{Spin}^{q}$ structure consisting of a principal $\mathrm{SO}(3)$ bundle $P_{\mathrm{SO}(3)}(M)$, a principal $\operatorname{Spin}^{q}(n)$ bundle $P_{\mathrm{Spin}^{q}(n)}(M)$ and a $\mathrm{Spin}^{q}$ equivariant projection map

$$
\xi: P_{\mathrm{Spin}^{q}(n)}(M) \longrightarrow P_{\mathrm{SO}(n)}(M) \times P_{\mathrm{SO}(3)}(M) .
$$

In particular, $M$ admits a $\operatorname{Spin}^{q}$ structure if and only if $w_{2}(M)=w_{2}\left(P_{\mathrm{SO}(3)}(M)\right)$. We refer the reader to [5] for the theory of Spin and $\mathrm{Spin}^{c}$ structures, and to [6] for $\mathrm{Spin}^{q}$ structures.

Let $E$ denote the rank 3 real vector bundle associated to $P_{\mathrm{SO}(3)}(M)$. Let $\Delta$ and $H$ denote the locally defined spinor bundles of $T M$ and $E$ respectively, where $H_{x} \cong \mathbb{H} \cong \mathbb{C}^{2}$ is isomorphic to the quaternions. If $M$ is not spin, the quaternionic spinor bundle $\Delta^{q}=\Delta \otimes H$ is globally defined. Otherwise, if $M$ is Spin, we can use the trivial Spin ${ }^{q}$ structure. Since $\operatorname{Sp}(1)=\operatorname{Spin}(3)=\mathrm{SU}(2)$, we can consider the representations of $\mathrm{SU}(2)$ given by the symmetric tensor powers $S^{k} H$ of dimension $k+1$. From the definition of $\operatorname{Spin}^{q}$ structure, we see that $w_{2}(M)=w_{2}\left(S^{2} H\right)$, so that

- if $w_{2}(M)=0, M$ is Spin and we can take the tensor product of $\Delta$ with the symmetric powers $S^{2 k} H$, for $k \geqslant 0$;
- if $w_{2}(M) \neq 0, M$ is not Spin, but we can still take the tensor product of $\Delta$ with the symmetric powers $S^{2 k+1} H$ to get globally defined vector bundles, for $k \geqslant 0$.


## 3. Twisted Dirac operators and vanishing theorem

Let $M$ be an $2 n$-dimensional oriented Riemannian manifold admitting a $\operatorname{Spin}^{q}$ structure $P_{\text {Spin }^{q}(n)}(M)$. The Levi-Civita connection $\omega$ on $M$ together with a chosen fixed connection $\theta$ on $P_{\mathrm{SO}(3)}(M)$ define a connection on $P_{\text {Spin }^{q}(n)}(M)$ denoted by $\nabla^{q}$. The twisted Dirac operator $\not \partial \otimes S^{k} H$ is thus defined by

$$
\left(\not \supset \otimes S^{k} H\right)(\psi)=\sum_{i=1}^{n} v_{i} * \nabla_{v_{i}}^{q} \psi,
$$

where $*$ denotes Clifford multiplication, $\psi \in \Gamma\left(\Delta \otimes S^{k} H\right)$ and $\nabla^{q}$ also denotes the extension of the covariant derivative to the bundles $\Delta \otimes S^{k} H$.

By the Atiyah-Singer index theorem, the index of the twisted Dirac operators can be computed as

$$
\operatorname{ind}\left(\not \partial \otimes S^{k} H\right)=\left\langle\operatorname{ch}\left(S^{k} H\right) \hat{A}(M),[M]\right\rangle
$$

where $\operatorname{ch}(\cdot)$ denotes the Chern character, $\hat{A}(M)$ denotes the $\hat{A}$-genus belonging to the characteristic power series

$$
\frac{x / 2}{\sinh (x / 2)}=\frac{x}{\mathrm{e}^{x / 2}-\mathrm{e}^{-x / 2}},
$$

and [ $M$ ] denotes the fundamental cycle of $M$. Since $H$ is a rank 2 vector bundle such that $H \equiv H^{*}$, locally (by the splitting principle) it can be viewed as $H=\mathcal{L}^{1 / 2} \oplus \mathcal{L}^{-1 / 2}$, so that its Chern class and character are $c(H)=$ $(1+l / 2)(1-l / 2)$ and $\operatorname{ch}(H)=\mathrm{e}^{l / 2}+\mathrm{e}^{-l / 2}$ respectively, where $l / 2$ is a formal root. In terms of formal roots, if $c\left(T M_{c}\right)=\left(1+x_{1}\right)\left(1-x_{1}\right) \cdots\left(1+x_{n}\right)\left(1-x_{n}\right)$ and $p(T M)=\left(1+x_{1}^{2}\right) \cdots\left(1+x_{n}^{2}\right)$, then

$$
\operatorname{ind}\left(\not \partial \otimes S^{k} H\right)=\left\langle\left(\mathrm{e}^{k l / 2}+\mathrm{e}^{(k-2) l / 2}+\cdots+\mathrm{e}^{-(k-2) l / 2}+\mathrm{e}^{-k l / 2}\right) \prod_{i=1}^{n} \frac{x_{i}}{\mathrm{e}^{x_{i} / 2}-\mathrm{e}^{-x_{i} / 2}},[M]\right\rangle
$$

For the sake of simplicity in the notation, we shall use the Adams operators $\Psi^{k}$ on $H$ instead of the symmetric powers [3], since $\operatorname{ch}\left(\Psi^{k} H\right)=\mathrm{e}^{k l / 2}+\mathrm{e}^{-k l / 2}$. In fact, by the splitting principle $S^{k} H=\Psi^{k} H+S^{k-2} H$, so that it is equivalent to deal with these virtual vector bundles.

If $M$ admits a non-trivial $S^{1}$ action compatible with the $\operatorname{Spin}^{q}$ structure, the equivariant version of the index can be written in terms of the local data of the $S^{1}$-fixed point set $M^{S^{1}}$. More precisely, let $z \in S^{1}$ be a generic element of $S^{1}$. By the Atiyah-Singer fixed point theorem [2]

$$
\operatorname{ind}\left(\not \partial \otimes \Psi^{k} H\right)_{z}=\sum_{P \subset M^{S^{1}}} \mu(P, z)
$$

where $\mu(P, z)$ is the local contribution of the fixed point component (submanifold) $P \subset M^{S^{1}}$, which can be computed as follows. The $S^{1}$ action on $M$ induces a decomposition of $T M$ over $P$

$$
\begin{equation*}
\left.T M\right|_{P}=L^{m_{1}}+\cdots+L^{m_{n}} \tag{1}
\end{equation*}
$$

where $L$ corresponds to the standard representation of $S^{1}$, so that $z \in S^{1}$ acts by multiplication by $z^{m_{i}}$ on $L^{m_{i}}$. The integers $m_{i}=m_{i}(P) \in \mathbb{Z}$ are the exponents of the action at $P$ whose signs can be changed in pairs. Thus,

$$
\mu(P, z)=\left\langle\left(z^{-k h / 2} \mathrm{e}^{k l / 2}+z^{k h / 2} \mathrm{e}^{-k l / 2}\right) \prod_{m_{i}=0} \frac{x_{i}}{\mathrm{e}^{x_{i} / 2}-\mathrm{e}^{-x_{i} / 2}} \prod_{m_{j} \neq 0} \frac{1}{z^{-m_{j} / 2} \mathrm{e}^{x_{j} / 2}-z^{m_{j} / 2} \mathrm{e}^{-x_{j} / 2}},[P]\right\rangle,
$$

where $h=h(P)$ is the exponent of the action on the auxiliary rank 3 bundle $E$ restricted to $P$. Furthermore, we can consider $\mu(P, z)=\mu_{+}(P, z)+\mu_{-}(P, z)$ where

$$
\begin{aligned}
& \mu_{+}(P, z)=\left\langle z^{-k h / 2} \mathrm{e}^{k l / 2} \prod_{m_{i}=0} \frac{x_{i}}{\mathrm{e}^{x_{i} / 2}-\mathrm{e}^{-x_{i} / 2}} \prod_{m_{j} \neq 0} \frac{1}{z^{-m_{j} / 2} \mathrm{e}^{x_{j} / 2}-z^{m_{j} / 2} \mathrm{e}^{-x_{j} / 2}},[P]\right\rangle, \\
& \mu_{-}(P, z)=\left\langle z^{k h / 2} \mathrm{e}^{-k l / 2} \prod_{m_{i}=0} \frac{x_{i}}{\mathrm{e}^{x_{i} / 2}-\mathrm{e}^{-x_{i} / 2}} \prod_{m_{j} \neq 0} \frac{1}{z^{-m_{j} / 2} \mathrm{e}^{x_{j} / 2}-z^{m_{j} / 2} \mathrm{e}^{-x_{j} / 2}},[P]\right\rangle
\end{aligned}
$$

$\mu_{+}(P, z)$ and $\mu_{-}(P, z)$ are both rational functions of the complex variable $z$ with zeroes at 0 and $\infty$ as long as

$$
|k h(P)|<\left|m_{1}(P)\right|+\cdots+\left|m_{n}(P)\right|
$$

If such a condition is fulfilled for all $P \subset M^{S^{1}}$, then $f(z)=\operatorname{ind}\left(\not \partial \otimes \Psi^{k} H\right)_{z}$ is a rational function of $z$ with $f(0)=0$ and $f(\infty)=0$. Notice that $\operatorname{ind}\left(\not \partial \otimes \Psi^{k} H\right)_{z}$ also belongs to the representation ring $R\left(S^{1}\right)$ of $S^{1}$ which can be identified with the Laurent polynomial ring $\mathbb{Z}\left[z, z^{-1}\right]$. Hence, $f(z)$ must be identically zero and

$$
\operatorname{ind}\left(\not \partial \otimes \Psi^{k} H\right)=\operatorname{ind}\left(\not \partial \otimes \Psi^{k} H\right)_{1}=0
$$

Thus we have proved the following:

Theorem 3.1. Let $M$ be a $\operatorname{Spin}^{q}$ manifold admitting a non-trivial smooth $S^{1}$ action lifting to the $\operatorname{Spin}^{q}$ structure. Let $H$ denote the (locally defined) rank 2 vector bundle given by the $\operatorname{Spin}^{q}$ structure. Let $k_{0} \in \mathbb{Z}$ be the greatest non-negative integer such that the twisted Dirac operator $\not \partial \otimes S^{k_{0}} H$ is well-defined and

$$
\left|k_{0} h(P)\right|<\left|m_{1}(P)\right|+\cdots+\left|m_{n}(P)\right|
$$

for all components $P \subset M^{S^{1}}$, where $h(P)$ and $m_{i}(P)$ denote the exponents of the $S^{1}$ actions on the auxiliary bundle $P_{\mathrm{SO}(3)}$ and the tangent bundle $T M$ restricted to $P$ respectively. Then

$$
\left\langle\operatorname{ch}\left(\hat{A}(M) S^{k} H\right),[M]\right\rangle=0
$$

for all $0 \leqslant k \leqslant k_{0}$ and $k \equiv k_{0}(\bmod 2)$.

## 4. Almost quaternionic manifolds

A $4 n$-dimensional manifold $M, n>1$, is called almost quaternionic if there is a rank 3 sub-bundle $Q$ of the endomorphism bundle $\operatorname{End}(T M)=T^{*} M \otimes T M$ such that locally $Q$ has an (admissible) basis $\{I, J, K\}$ satisfying the relations $I^{2}=J^{2}=-1$ and $K=I J,-J I$. The existence of the sub-bundle $Q$ implies the reduction of structure of the frame bundle of $M$ to a sub-bundle $P$ with structure group $\mathrm{Gl}_{n}(\mathbb{H}) \times \mathbb{Z}_{2} \mathrm{Sp}(1) \subset \mathrm{Gl}_{4 n}(\mathbb{R})$. Thus, the complexified tangent bundle of $M$ has the form

$$
\begin{equation*}
T M_{c}=E \otimes H, \tag{2}
\end{equation*}
$$

where $E$ and $H$ correspond to the standard complex representations $\mathbb{C}^{2 n}$ and $\mathbb{C}^{2}$ of $\mathrm{Gl}_{n}(\mathbb{H})$ and $\mathrm{Sp}(1)$ respectively. The bundle $Q$ naturally endows this type of manifold with a $\operatorname{Spin}^{q}$ structure [6].

Theorem 4.1. Let $M$ be a 4n-dimensional almost quaternionic manifold admitting a (non-trivial) smooth $S^{1}$ action preserving the almost quaternionic structure $Q$. Then

$$
\left\langle\operatorname{ch}\left(S^{k} H\right) \hat{A}(M),[M]\right\rangle=0
$$

for all $0 \leqslant k<n$ with $k \equiv n(\bmod 2)$, where $S^{k} H$ denotes the $k$-th symmetric power of the factor $H$ of $T M_{c}$ described above.

Proof. First, choose a Riemannian metric on $M$ and average it over the circle action. Using this metric, define a quaternionic-Hermitian metric in the usual way. With these choices, the manifold $M$ is now an almost quaternionHermitian with a compatible isometric circle action. Its structure group reduces to $\operatorname{Sp}(n) \operatorname{Sp}(1)$, so that $H \cong H^{*}$ and $E \cong E^{*}$. Since the $S^{1}$ action preserves the $\operatorname{Sp}(n) \operatorname{Sp}(1)$ structure, it preserves the bundles associated to it, such as $H \otimes H, E \otimes E$.

Let $P \subset M^{S^{1}}$ be an $S^{1}$-fixed submanifold. Since $S^{1}$ consists of automorphisms of the $\operatorname{Sp}(n) \operatorname{Sp}(1)$ structure, the locally defined bundles $H$ and $E$ also split along $P$

$$
\left.H\right|_{P}=L^{h / 2}+L^{-h / 2},\left.\quad E\right|_{P}=L^{e_{1} / 2}+L^{-e_{1} / 2}+\cdots+L^{e_{n} / 2}+L^{-e_{n} / 2}
$$

with integers $e_{j}=e_{j}(P) \in \mathbb{Z}$ and $h=h(P) \in \mathbb{Z}$. By (1) and (2),

$$
\pm m_{i}=\frac{e_{a}+h}{2} \quad \text { or } \quad \pm m_{i}=\frac{e_{a}-h}{2}
$$

for some $1 \leqslant a \leqslant n$. Thus, the inequality of Theorem 3.1 is

$$
|k h|<\frac{1}{2} \sum_{i=1}^{n}\left(\left|h+e_{i}\right|+\left|h-e_{i}\right|\right) .
$$

If we choose the signs of the exponents $m_{i}$ appropriately, we have $k|h|<n|h|$, i.e. $k<n$.

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