Available online at www.sciencedirect.com
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C. R. Acad. Sci. Paris, Ser. I 345 (2007) 199–202
COMPTES RENDUS MATHEMATIQUE
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Ordinary Differential Equations

Periodic solutions for a discrete time predator–prey system with monotone functional responses

Mostafa Fazly, Mahmoud Hesaaraki

Department of Mathematical Sciences, Sharif University of Technology, P.O. Box 11365-9415, Azadi Ave., Tehran, Iran

Received 23 October 2006; accepted after revision 5 June 2007

Available online 31 July 2007

Presented by Jean-Michel Bony

Abstract

In this Note, sharp sufficient conditions for the existence of periodic solutions of a nonautonomous discrete time semi-ratio-dependent predator–prey system with functional responses are derived. In our results this system with any monotone functional response bounded by polynomials in $\mathbb{R}^+$, always has at least one ω-periodic solution. In particular, this system with the most popular functional responses Michaelis–Menten, Holling type-II and III, sigmoidal, Ivlev and some other monotone response functions, always has at least one ω-periodic solution. ***To cite this article: M. Fazly, M. Hesaaraki, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Solutions périodiques d'un système discret prédateur–proie du temps avec des réponses fonctionnelles monotones. Dans cette Note, on donne de nouvelles conditions suffisantes d'existence d'une solution périodique d'un système discret non autonome prédateur–proie, dépendant du temps avec condition semi-finale et pour différentes réponses fonctionnelles. Dans les résultats obtenus, pour des réponses fonctionnelles monotones majorées par des polynômes sur $\mathbb{R}^+$, le système a toujours au moins une solution ω-périodique. En particulier, pour les réponses fonctionnelles les plus utilisées – Michaelis–Menten, type-II et III de Holling, sigmoïde, Ivlev et d'autres fonctions monotones – le système a toujours une solution ω-périodique. ***Pour citer cet article : M. Fazly, M. Hesaaraki, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction

In the past decades, many authors have studied the existence of periodic solutions for the differential and difference equations, [1–3,6]. In particular, the existence of periodic solutions of the following semi-ratio-dependent predator–prey system with some monotone functional responses has been studied extensively in the literature [1,4,6],

$$\begin{cases} \dot{x}_1(t) = \big(a(t) - b(t)x_1\big)x_1 - p(t, x_1)x_2, \\ \dot{x}_2(t) = \left(c(t) - d(t)\dfrac{x_2}{x_1}\right)x_2. \end{cases} \tag{1}$$

E-mail addresses: mostafafazly@yahoo.com, mostafafazly@math.sharif.edu (M. Fazly), hesaraki@sina.sharif.edu (M. Hesaaraki).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2007.06.021

It is well known that discrete time systems are more appropriate than continuous ones when the populations have nonoverlapping generations. In addition, discrete time models can also provide efficient computational systems of continuous for numerical simulations.

In this work, we shall study the existence of periodic solutions for the following nonautonomous discrete time system analogue of system (1) that is studied by Fan and Wang in [2], Bohner et al. in [1] and in other literature,

$$\begin{cases} x_1(k+1) = x_1(k)\exp\bigl(a(k) - b(k)x_1(k) - p\bigl(k, x_1(k)\bigr)x_2(k)/x_1(k)\bigr), \\ x_2(k+1) = x_2(k)\exp\bigl(c(k) - d(k)x_2(k)/x_1(k)\bigr), \end{cases} \tag{2}$$

where $a, c:\mathbb{Z}\to\mathbb{R}$ and $\bar{a}, \bar{c} > 0$, also $b, d:\mathbb{Z}\to\mathbb{R}^+$ are ω-periodic sequences of real and $k\in\mathbb{Z}$.

The most popular prey-dependent monotone functional responses which are studied extensively by authors are $p_1(k,x) = m(k)x$, that appeared in the classical Lotka–Volterra model, $p_2(k,x) = m(k)x/(A(k)+x)$ was proposed by Michaelis and Menten in the context of studying enzymatic reactions. Later, such a model was also used by Holling, now, popularly referred to as a Holling type-II response function. Another type, known as the Holling type-III functional response, takes the form $p_3(k,x) = m(k)x^2/(A(k)+x^2)$ and in general case, $p_4(k,x) = m(k)x^n/(A(k)+x^n)$, $n > 2$, is known as the sigmoidal response function. Similar types of celebrated response functions can be found in Freedman [4], e.g. $p_5(k,x) = m(k)x^2/((A(k)+x)(B(k)+x))$ and $p_6(k,x) = m(k)(1-\mathrm{e}^{-A(k)x})$ which was proposed by Ivlev. Throughout this Note, we assume that A, B and m are ω-periodic sequences of positive real numbers.

In this Note, using a new estimation technique and a continuation theorem from Gaines and Mawhin, we show that system (2) with all of the monotone functional responses bounded by polynomials in $\mathbb{R}^+$, always has at least one ω-periodic solution. In particular, system (2) with the most popular monotone functional responses p_1, called Leslie–Gower system, p_2, called Holling–Tanner system, p_3, p_4, p_5 and p_6 always has at least one ω-periodic solution. Our results extend previous works presented in [1,2] and [6].

2. Existence of positive periodic solutions

In this section we shall prove a theorem related to system (2). By this theorem, we get sharp sufficient conditions for the existence of periodic solutions of system (2) with the monotone functional responses.

For convenience, we use the following notations in our proofs:

$$I_\omega = \{0, 1, \ldots, \omega-1\}, \quad g^u = \max_{k\in I_\omega} g(k), \quad g^l = \min_{k\in I_\omega} g(k), \quad \bar{g} = 1/\omega\sum g(s), \quad \sum g(s) = \sum_{s\in I_\omega} g(s),$$

for $\{g(k)\}_{k\in\mathbb{Z}}$ an ω-periodic sequence of real numbers.

Obviously, for $k_1\in I_\omega$, $k\in\mathbb{Z}^+$ and ω-periodic function $f:\mathbb{Z}\to\mathbb{R}$ we have the following inequalities:

$$f(k) \leqslant f(k_1) + \sum\bigl|f(s+1) - f(s)\bigr| \quad\text{and}\quad f(k) \geqslant f(k_1) - \sum\bigl|f(s+1) - f(s)\bigr|. \tag{3}$$

Theorem 1. *Assume that in system* (2) *the following conditions hold*:

- (i) *The response function $p:\mathbb{Z}\times\mathbb{R}^+\to\mathbb{R}^+$ is ω-periodic with respect to the first variable and is differentiable with respect to the second variable, also $p(k,0) = 0$ and $\frac{\partial p}{\partial x}(k,x) > 0$, for all $k\in\mathbb{Z}$, $x\in\mathbb{R}^+$.*
- (ii) *There exist $m\in\mathbb{Z}^+$ and ω-periodic functions $a_i:\mathbb{Z}\to\mathbb{R}^+\cup\{0\}$, $i = 0, \ldots, m-1$, such that $p(k,x) \leqslant a_0(k)x^m + \cdots + a_{m-1}(k)x$, for all $k\in\mathbb{Z}$ and $x\in\mathbb{R}^+$.*

Then this system has at least one positive ω-periodic solution.

Proof. First, let $x_i(k) = \exp(u_i(k))$, $i = 1, 2$, and $k\in\mathbb{Z}^+$. Using this change of variables, system (2) reduces to the following system:

$$\begin{cases} u_1(k+1) - u_1(k) = a(k) - b(k)\exp\bigl(u_1(k)\bigr) - p\bigl(k, \exp\bigl(u_1(k)\bigr)\bigr)\exp\bigl(u_2(k) - u_1(k)\bigr), \\ u_2(k+1) - u_2(k) = c(k) - d(k)\exp\bigl(u_2(k) - u_1(k)\bigr). \end{cases} \tag{4}$$

Here we would like to apply a continuation theorem from Gaines and Mawhin in [5, p. 40]; to do this, let

$$X = Y = \bigl\{u = \bigl\{u(k)\bigr\} \bigm| u(k)\in\mathbb{R}^2, u(k+\omega) = u(k), k\in\mathbb{Z}^+\bigr\} \quad\text{and}\quad \|u\| = \sum_{i=1,2}\max_{k\in I_\omega}\bigl|u_i(k)\bigr|, \quad u\in X.$$

Then X and Y with the above norm $\|\cdot\|$, are Banach spaces. For $u \in X$ and $k \in \mathbb{Z}^+$, we define: $(Lu)(k)$ as the left-hand side of system (4), $(Nu)(k)$ as the right-hand side of system (4) and $Pu = Qu = \bar{u}$.

It is not difficult to show that $\dim \operatorname{Ker} L = 2 = \operatorname{codim} \operatorname{Im} L$. Since $\operatorname{Im} L$ is closed in Y thus L is a Fredholm mapping of index zero. It is easy to show that P and Q are continuous projections and $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. By the Arzela–Ascoli theorem, it can be shown that N is L-compact on $\overline{\Omega}$ for every open bounded set, $\Omega \subset X$. Now we are in the position to search for an appropriate open bounded subset Ω for the application of Gaines–Mawhin's continuation theorem in [5, p. 40]. Suppose that $u = (u_1, u_2) \in X$ is a solution of the operator equation $Lu = \lambda Nu$, for a certain $\lambda \in (0, 1)$. By summing on two sides of this operator equation over the set I_ω, we obtain:

$$\bar{a}\omega = \sum \left[\frac{p(k, \exp(u_1(k)))}{\exp(u_1(k))} \exp\bigl(u_2(k)\bigr) + b(k) \exp\bigl(u_1(k)\bigr) \right] \quad \text{and} \quad \bar{c}\omega = \sum \left[d(k) \frac{\exp(u_2(k))}{\exp(u_1(k))} \right]. \tag{5}$$

It follows from the operator equation $Lu = \lambda Nu$ and (5),

$$\sum \bigl| u_1(k+1) - u_1(k) \bigr| < \bigl(\bar{a} + \overline{|a|}\bigr)\omega \quad \text{and} \quad \sum \bigl| u_2(k+1) - u_2(k) \bigr| < \bigl(\bar{c} + \overline{|c|}\bigr)\omega. \tag{6}$$

Since $u = (u_1, u_2) \in X$, there exist $\xi_i, \eta_i \in I_\omega$, $i = 1, 2$, such that

$$u_i(\xi_i) = u_i^l \quad \text{and} \quad u_i(\eta_i) = u_i^u. \tag{7}$$

From (5) and (7) we obtain, $\bar{a}\omega \geqslant \sum b(k) \exp(u_1(\xi_1)) = \bar{b}\omega \exp(u_1(\xi_1))$. From this inequality, (6) and (3), for all $k \in \mathbb{Z}^+$ it follows

$$u_1(k) \leqslant u_1(\xi_1) + \sum \bigl| u_1(s+1) - u_1(s) \bigr| \leqslant \ln(\bar{a}/\bar{b}) + \bigl(\bar{a} + \overline{|a|}\bigr)\omega := M_1. \tag{8}$$

From (5) and (7) we have

$$\bar{c}\omega = \sum \bigl[d(k) \exp\bigl(u_2(k)\bigr) / \exp\bigl(u_1(k)\bigr) \bigr] \geqslant \bar{d}\, \omega \exp\bigl(u_2(\xi_2)\bigr) / \exp\bigl(u_1(\eta_1)\bigr), \tag{9}$$

then using (8) we also have, $u_2(\xi_2) \leqslant \ln(\bar{c}/\bar{d}) + u_1(\eta_1) \leqslant \ln(\bar{c}/\bar{d}) + M_1$. Again from this, (3) and (6), for all $k \in \mathbb{Z}^+$, we get

$$u_2(k) \leqslant u_2(\xi_2) + \sum \bigl| u_2(s+1) - u_2(s) \bigr| \leqslant \ln(\bar{c}/\bar{d}) + M_1 + \bigl(\bar{c} + \overline{|c|}\bigr)\omega := M_2. \tag{10}$$

It follows from (5) and (7) and assumption (ii) that

$$\bar{a} \leqslant \bar{b} \exp\bigl(u_1(\eta_1)\bigr) + \bigl(\bar{a}_0 \exp\bigl(u_1(\eta_1)\bigr)^{m-1} + \cdots + \bar{a}_{m-2} \exp\bigl(u_1(\eta_1)\bigr) + \bar{a}_{m-1}\bigr) \exp\bigl(u_2(\eta_2)\bigr). \tag{11}$$

In order to obtain M_3 and M_4 such that $u_1(k) \geqslant M_3$ and $u_2(k) \geqslant M_4$, for all $k \in \mathbb{Z}^+$, we consider two cases.

Case 1: If $\exp(u_1(\eta_1)) \leqslant \exp(u_2(\eta_2))$ then it follows from (3), (6), (8) and (11), for all $k \in \mathbb{Z}^+$, that

$$u_2(k) \geqslant \ln\bigl(\bar{a} / \bigl(\bar{b} + \bar{a}_{m-1} + \bar{a}_{m-2} \exp(M_1) + \cdots + \bar{a}_0 \exp(M_1)^{m-1}\bigr)\bigr) - (\bar{c} + \overline{|c|})\omega := M_4^1. \tag{12}$$

From this, (9) and (12) we get $u_1(\eta_1) \geqslant M_4^1 - \ln(\bar{c}/\bar{d})$, from this and (3), for all $k \in \mathbb{Z}^+$ we obtain

$$u_1(k) \geqslant u_1(\eta_1) - \sum \bigl| u_1(s+1) - u_1(s) \bigr| \geqslant M_4^1 - \ln(\bar{c}/\bar{d}) - \bigl(\bar{a} + \overline{|a|}\bigr)\omega := M_3^1. \tag{13}$$

Case 2: If $\exp(u_1(\eta_1)) > \exp(u_2(\eta_2))$ then it follows from (11) that

$$\bar{a} \leqslant (\bar{b} + \bar{a}_{m-1}) \exp\bigl(u_1(\eta_1)\bigr) + \bar{a}_{m-2} \exp\bigl(u_1(\eta_1)\bigr)^2 + \cdots + \bar{a}_0 \exp\bigl(u_1(\eta_1)\bigr)^m. \tag{14}$$

Now, consider the function $g : \mathbb{R}^+ \to \mathbb{R}$ with $g(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_{m-1} x - b_m$ such that b_i, $i = 0, \ldots, m-2$ are nonnegative and b_{m-1}, b_m are positive real numbers and $m \in \mathbb{Z}^+$. Obviously, the function g is increasing and $g(0) = -b_m < 0$, so g has a unique root x^* in $\mathbb{R}^+$. Then for $x \in \mathbb{R}^+$ and $g(x) \geqslant 0$, we have $x \geqslant x^*$. Hence it follows from (14) that $\exp(u_1(\eta_1)) \geqslant x^*$, where x^* only depends on $\bar{a}, \bar{b}, m$ and $\bar{a}_i$, $i = 0, \ldots, m-1$. From this, (6) and (3), for all $k \in \mathbb{Z}^+$ we obtain

$$u_1(k) \geqslant u_1(\eta_1) - \sum \bigl| u_1(s+1) - u_1(s) \bigr| \geqslant \ln(x^*) - \bigl(\bar{a} + \overline{|a|}\bigr)\omega := M_3^2. \tag{15}$$

From (5) and (7) we have, $\bar{c}\omega = \sum [d(k) \exp(u_2(k)) / \exp(u_1(k))] \leqslant \exp(u_2(\eta_2)) / \exp(u_1(\xi_1))\, \bar{d}\omega$.

It follows from this and (15) that $u_2(\eta_2) \geqslant \ln(\bar{c}/\bar{d}) + M_3^2$. From this, (6) and (3), for all $k \in \mathbb{Z}^+$, we obtain:

$$u_2(k) \geqslant u_2(\eta_2) - \sum \left|u_2(s+1) - u_2(s)\right| \geqslant \ln(\bar{c}/\bar{d}) + M_3^2 - \left(\bar{c} + \overline{|c|}\right)\omega := M_4^2. \tag{16}$$

Now we take $M_3 := \min\{M_3^1, M_3^2\}$ and $M_4 := \min\{M_4^1, M_4^2\}$. Then for all $k \in \mathbb{Z}^+$, it follows from (12), (13), (15) and (16) that $u_1(k) \geqslant M_3$ and $u_2(k) \geqslant M_4$. Hence from this, (8) and (10) we have $\max_{k \in I_\omega} |u_1(k)| \leqslant \max\{|M_1|, |M_3|\} := R_1$ and $\max_{k \in I_\omega} |u_2(k)| \leqslant \max\{|M_2|, |M_4|\} := R_2$. Clearly R_1 and R_2 do not depend on λ. In order to define Ω and to show that $QN(u) \neq 0$ for $u \in \partial\Omega \cap \operatorname{Ker} L$, we prove that the following algebraic equations have a unique solution in $\mathbb{R}_+^2$;

$$\bar{a} - \bar{b}x_1 - x_2\bar{p}(x_1)/x_1 = 0, \qquad \bar{c} - \bar{d}x_2/x_1 = 0, \tag{17}$$

where $\bar{p}(x_1) = 1/\omega \sum_{k=0}^{\omega-1} p(k, x_1)$. From the assumptions upon a, b, c, d and p, it follows that the reduced equation in x_1, i.e. $\bar{a} - \bar{b}x_1 - \bar{p}(x_1)\bar{c}/\bar{d} = 0$ has a unique solution x_1^* in $\mathbb{R}^+$. So, (17) has the unique solution (x_1^*, x_2^*) with $x_1^* > 0, x_2^* > 0$. In fact, we can choose $R > R_1 + R_2$ sufficiently large such that $\|(\ln(x_1^*), \ln(x_2^*))\| = |\ln(x_1^*)| + |\ln(x_2^*)| < R$.

Now we take $\Omega := \{u \in X \mid \|u\| < R\}$. This satisfies the first condition of the Gaines–Mawhin's continuation theorem. Let $u \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap \mathbb{R}^2$, then u is a constant vector in $\mathbb{R}^2$ with $\sum_1^2 |u_i| = R$. Hence, from (17) and the definition of R, we see that $QN(u) \neq 0$.

Finally, by direct calculation and from the assumptions upon b, c and p, and also since $x_1^*, x_2^* > 0$, we have:

$$\deg(JQN, \Omega \cap \operatorname{Ker} L, 0) = \operatorname{sgn}\left\{\bar{c}\left(\bar{b}x_1^* + \frac{\partial \bar{p}}{\partial x_1}(x_1^*)\, x_2^*\right)\right\} > 0,$$

where $\deg(\cdot,\cdot,\cdot)$ is the Brouwer degree and the isomorphism J can be chosen to be the identity mapping, since $\operatorname{Im} Q = \operatorname{Ker} L$. Hence the second condition of the Gaines–Mawhin's continuation theorem holds, too. Thus system (4) has at least one ω-periodic solution $u^* = \{(u_1^*(k), u_2^*(k))\}_{k \in \mathbb{Z}^+}$, we see that $x^* = \{(\exp(u_1^*(k)), \exp(u_2^*(k)))\}_{k \in \mathbb{Z}^+}$ is a positive ω-periodic solution of system (2). □

Remark 1. The interesting point is that for the functional responses $p_1, \ldots, p_6$ and some more popular functional responses that are used by some authors, the assumptions of Theorem 1 hold. In fact, if n is the number in p_4 and if $m = n$, $a_0 = a_{n-2} = m/A$, $a_{n-1} = m(1 + A + 1/A + 1/(A+B))$ and $a_1 = \cdots = a_{n-3} = 0$ in (ii), then we see that the system (2) with $p_1, \ldots, p_5$ and p_6 satisfies the assumptions of Theorem 1.

Remark 2. For the existence of periodic solutions of system (2) with p_1, p_2, p_5 and p_6, Fan and Wang in [2] and Bohner et al. in [1] get Theorem 2.1 in [2] and Theorem 3.4 in [1]. In fact, Theorem 1 implies these theorems without using the exponential condition $\bar{p}_0\bar{c}\exp((\bar{a} + \overline{|a|} + \bar{c} + \overline{|c|})\omega) < \bar{b}\bar{d}$ which is used by them in [1] and [2]. Also, for system (2) with $p_2, \ldots, p_5$ and p_6, they have used the condition $d^l\bar{a} > p_1^u\bar{c}$, in Theorem 2.2 in [2] and Theorem 3.5 in [1], as we see in Theorem 1 this condition is not necessary.

Remark 3. By a similar discussion as in the proof of Theorem 1, one can get the same results for the continuous time system (1) with the monotone functional responses, which extend and improve some previous results presented in [1,6] and in other literature.

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