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Partial Differential Equations

Degenerate anisotropic variational inequalities with L^1-data

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Abstract

In this Note we introduce notions of T-solution and shift T-solution of variational inequalities corresponding to a nonlinear degenerate anisotropic elliptic operator, a set of constraints of a sufficiently large class and an L^1-right-hand side. We give theorems on the existence, uniqueness and properties of these solutions and describe their relation with solutions of variational inequalities in usual sense. ***To cite this article: A.A. Kovalevsky, Y.S. Gorban, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Inéquations variationnelles dégénérescents anisotropes avec données dans L^1. Dans cette Note nous introduisons les notions de T-solution et T-solution translatante des inéquations variationnelles correspondant à un opérateur non linéair dégénérescent anisotrope elliptique, un ensemble de contraintes d'une classe suffisament large et le second membre dans L^1. Nous donnons les théorèmes d'existence, d'unicité et de propriétées de ces solutions et décrivons leur relation avec solutions des inéquations variationnelles au sens ordinaire. ***Pour citer cet article : A.A. Kovalevsky, Y.S. Gorban, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction

During the last 10–15 years, investigations on the existence and properties of solutions of nonlinear equations and variational inequalities with L^1-data or measures as data have been intensively developed. As is generally known, an effective approach to the solvability of second-order equations in divergence form with L^1-right-hand sides has been proposed in [3]. In this connection we also mention a series of other close researches for nonlinear second-order equations with L^1-data, measures and renormalized solutions [6–8,11]. As far as elliptic variational inequalities with the same peculiarities are concerned, the known results are obtained mainly with the use of the same approaches and techniques which 'work' in the case of equations. These results relate, for the most part, to the unilateral Dirichlet problem (see for instance [1,4,5,10,12,13]), and in most cases (except for [10,13]) it is assumed that the corresponding sets of constraints contain bounded functions. Solvability of nonlinear variational inequalities with L^1-right-hand sides

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and bilateral obstacles was studied in [2]. There it was also supposed that the corresponding set of constraints contains bounded functions. We observe that variational inequalities were considered in [4,5,10,12,13] for nondegenerate and in [1,2] for degenerate isotropic operators.

In the present Note we introduce notions of T-solution and shift T-solution of variational inequalities corresponding to a nonlinear degenerate anisotropic elliptic operator, a set of constraints of a sufficiently large class and an L^1-right-hand side. We give theorems on the existence and uniqueness of these solutions and describe some of their properties. The notion of T-solution is connected with the condition that the set of constraints under consideration contains bounded functions, and the notion of shift T-solution does not require this condition. We state some assertions on relations between the kinds of solutions we introduce and solutions of variational inequalities in usual sense. The main results of the work are established under minimal conditions on the weighted functions involved. These results are new not only because they combine such features as degeneration and anisotropy, but also because they can be applied to a larger class of constraints as compared with those considered before. Indeed, our main assumption on the set of constraints as particular cases admits constraints defined by unilateral and bilateral obstacles.

The full paper with complete proofs of the results given in this Note is prepared in Russian for publication as a preprint of the institute where the first author works. Moreover, it will be submitted to a journal which is translated in English.

2. Preliminaries

Let Ω be a bounded open set of $\mathbb{R}^n$ $(n \geqslant 2)$, and let for every $i \in \{1, \ldots, n\}$, $q_i \in (1, n)$ and ν_i be a nonnegative function on Ω such that $\nu_i > 0$ a.e. in Ω, $\nu_i \in L^1_{\mathrm{loc}}(\Omega)$ and $(1/\nu_i)^{1/(q_i-1)} \in L^1(\Omega)$. We set $q = \{q_i\colon i = 1, \ldots, n\}$, $\nu = \{\nu_i\colon i = 1, \ldots, n\}$ and denote by $W^{1,q}(\nu, \Omega)$ the set of all functions $u \in L^1(\Omega)$ such that for every $i \in \{1, \ldots, n\}$ there exists the weak derivative $D_i u$ and $\nu_i |D_i u|^{q_i} \in L^1(\Omega)$. $W^{1,q}(\nu, \Omega)$ is a Banach space with the norm $\|u\|_{1,q,\nu} = \int_\Omega |u|\,\mathrm{d}x + \sum_{i=1}^n (\int_\Omega \nu_i |D_i u|^{q_i}\,\mathrm{d}x)^{1/q_i}$. We denote by $W^{1,q}_0(\nu, \Omega)$ the closure of $C^\infty_0(\Omega)$ in $W^{1,q}(\nu, \Omega)$.

Let for every $k > 0$, $T_k : \mathbb{R} \to \mathbb{R}$ be the function defined by $T_k(s) = \max\{\min\{s, k\}, -k\}$. By $\mathcal{T}^{1,q}_0(\nu, \Omega)$ we denote the set of all functions $u : \Omega \to \mathbb{R}$ such that for every $k > 0$, $T_k(u) \in W^{1,q}_0(\nu, \Omega)$.

Let $c_1, c_2 > 0$, $g_1, g_2 \in L^1(\Omega)$, $g_1, g_2 \geqslant 0$ in Ω, and let for every $i \in \{1, \ldots, n\}$, $a_i : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function. We suppose that for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$,

$$\sum_{i=1}^n (1/\nu_i)^{1/(q_i-1)}(x)\big|a_i(x,\xi)\big|^{q_i/(q_i-1)} \leqslant c_1 \sum_{i=1}^n \nu_i(x)|\xi_i|^{q_i} + g_1(x),$$

$$\sum_{i=1}^n a_i(x,\xi)\xi_i \geqslant c_2 \sum_{i=1}^n \nu_i(x)|\xi_i|^{q_i} - g_2(x).$$

Moreover, we assume that for almost every $x \in \Omega$ and every $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$,

$$\sum_{i=1}^n \big[a_i(x,\xi) - a_i(x,\xi')\big](\xi_i - \xi'_i) > 0.$$

Let $\mathcal{A} : W^{1,q}_0(\nu, \Omega) \to (W^{1,q}_0(\nu, \Omega))^*$ be the operator such that for every $u, v \in W^{1,q}_0(\nu, \Omega)$,

$$\langle \mathcal{A}u, v\rangle = \int_\Omega \left\{\sum_{i=1}^n a_i(x, \nabla u) D_i v\right\} \mathrm{d}x.$$

3. *T*-solutions of variational inequalities

Let V be a closed convex set in $W^{1,q}_0(\nu, \Omega)$ satisfying the conditions: $V \cap L^\infty(\Omega) \neq \emptyset$ and

$$\text{if } u, v \in V \text{ and } k > 0, \text{ then } u - T_k(u - v) \in V. \qquad (*)$$

Definition 1. Let $f \in L^1(\Omega)$. A T-solution of the variational inequality corresponding to the triplet $(\mathcal{A}, V, f)$ is a function $u \in \mathcal{T}_0^{1,q}(\nu, \Omega)$ such that:

(i) for every $v \in V \cap L^\infty(\Omega)$ and $k \geqslant 1$ we have $v - T_k(v-u) \in V$;
(ii) if $v \in V \cap L^\infty(\Omega)$, $k \geqslant 1$ and $k_1 = k + \|v\|_{L^\infty(\Omega)}$, then $\langle \mathcal{A}T_{k_1}(u), T_k(u-v)\rangle \leqslant \int_\Omega f T_k(u-v)\,\mathrm{d}x$.

Theorem 2. *Let $f \in L^1(\Omega)$. Then there exists a unique T-solution of the variational inequality corresponding to the triplet $(\mathcal{A}, V, f)$.*

Proving this result, in general outline we follow the approach of [3] and use some ideas of [9].
We set $\bar{q} = n(\sum_{i=1}^n \frac{1}{q_i})^{-1}$.

Theorem 3. *Let $m \in \mathbb{R}^n$, and $m_i > 0$, $i = 1, \ldots, n$. Suppose that for every $i \in \{1, \ldots, n\}$, $1/\nu_i \in L^{m_i}(\Omega)$ and $\frac{1}{m_i} + \frac{\bar{q}}{n(\bar{q}-1)} \sum_{j=1}^n \frac{1}{m_j q_j} < q_i - \frac{(n-1)\bar{q}}{n(\bar{q}-1)}$. Let $f \in L^1(\Omega)$ and u be the T-solution of the variational inequality corresponding to the triplet $(\mathcal{A}, V, f)$. Then $u \in W_0^{1,1}(\Omega)$.*

Remark 4. In the nondegenerate isotropic case ($q_i = q_1$, $\nu_i = 1$, $i = 1, \ldots, n$) the conditions of Theorem 3 on q_i, m_i and ν_i are reduced to the well known requirement $q_1 > 2 - 1/n$. In the general case, from these conditions it follows that $\min\{q_i\colon i = 1, \ldots, n\} > \frac{(n-1)\bar{q}}{n(\bar{q}-1)}$, and examples show that the violation of the latter inequality may lead to the violation of the belonging of T-solutions of variational inequalities to $W_0^{1,1}(\Omega)$. The mentioned inequality for the minimum of the numbers q_i has already been used in [7] in the study of the solvability of nondegenerate anisotropic equations with measure data.

For every $m \in \mathbb{R}^n$ such that $m_i > 0$, $i = 1, \ldots, n$, we set $p_m = n(\sum_{i=1}^n \frac{1+m_i}{m_i q_i} - 1)^{-1}$.

Proposition 5. *Suppose that $m \in \mathbb{R}^n$ and the following condition is satisfied: for every $i \in \{1, \ldots, n\}$, $m_i \geqslant 1/(q_i - 1)$ and $1/\nu_i \in L^{m_i}(\Omega)$. Let $f \in L^{p_m/(p_m-1)}(\Omega)$. Then u is the T-solution of the variational inequality corresponding to the triplet $(\mathcal{A}, V, f)$ if and only if $u \in V$ and for every function $v \in V$, $\langle \mathcal{A}u, u - v\rangle \leqslant \int_\Omega f(u-v)\,\mathrm{d}x$.*

4. Shift T-solutions of variational inequalities

Let V be a nonempty closed convex set in $W_0^{1,q}(\nu, \Omega)$ satisfying only condition $(*)$.

Definition 6. Let $f \in L^1(\Omega)$ and $\psi \in V$. A ψ-shift T-solution of the variational inequality corresponding to the triplet $(\mathcal{A}, V, f)$ is a function $u : \Omega \to \mathbb{R}$ such that:

(i) for every $v \in (-\psi + V) \cap L^\infty(\Omega)$ and $k \geqslant 1$ we have $v + \psi + T_k(u - v - \psi) \in V$;
(ii) if $v \in (-\psi + V) \cap L^\infty(\Omega)$, $k \geqslant 1$ and $k_1 = k + \|v\|_{L^\infty(\Omega)}$, then $\langle \mathcal{A}(\psi + T_{k_1}(u - \psi)), T_k(u - v - \psi)\rangle \leqslant \int_\Omega f T_k(u - v - \psi)\,\mathrm{d}x$.

We observe that if $f \in L^1(\Omega)$, $\psi \in V$ and u is a ψ-shift T-solution of the variational inequality corresponding to the triplet $(\mathcal{A}, V, f)$, then for every function $\varphi \in V$ we have $u - \varphi \in \mathcal{T}_0^{1,q}(\nu, \Omega)$.

Theorem 7. *Let $f \in L^1(\Omega)$ and $\psi \in V$. Then there exists a ψ-shift T-solution of the variational inequality corresponding to the triplet $(\mathcal{A}, V, f)$.*

Theorem 8. *Let $f \in L^1(\Omega)$, $\psi, \varphi \in V$ and $\psi - \varphi \in L^\infty(\Omega)$. Let u_1 be a ψ-shift T-solution of the variational inequality corresponding to the triplet $(\mathcal{A}, V, f)$, and u_2 be a φ-shift T-solution of the variational inequality corresponding to the same triplet. Then $u_1 = u_2$ a.e. in Ω.*

The proof of Theorems 7 and 8 is based on the use of Theorem 2.

Proposition 9. *Let all the assumptions of Proposition* 5 *hold, and let* $\psi \in V$. *Then* u *is the* ψ*-shift* T*-solution of the variational inequality corresponding to the triplet* $(\mathcal{A}, V, f)$ *if and only if* $u \in V$ *and for every function* $v \in V$, $\langle \mathcal{A}u, u - v\rangle \leqslant \int_\Omega f(u - v)\,\mathrm{d}x$.

Proving Propositions 5 and 9 we use a Sobolev-type inequality in the anisotropic case (see [14, Th. 1.2]).

Remark 10. If $V \cap L^\infty(\Omega) \neq \emptyset$, $f \in L^1(\Omega)$ and $\psi \in V \cap L^\infty(\Omega)$, the T-solution and the ψ-shift T-solution of the variational inequality corresponding to the triplet $(\mathcal{A}, V, f)$ coincide.

Remark 11. The sets of functions $u \in W_0^{1,q}(\nu, \Omega)$ defined by one of the inequalities $u \geqslant w$, $u \leqslant w$ and $w \leqslant u \leqslant z$ a.e. in Ω with $w, z \in W_0^{1,q}(\nu, \Omega)$ satisfy condition $(*)$. Other simple examples of the sets satisfying condition $(*)$ are as follows: $V = \{u \in W_0^{1,q}(\nu, \Omega)\colon |\nabla u| \leqslant \varphi$ a.e. in $\Omega\}$, where φ is a nonnegative function on Ω, and $V = \{u \in W_0^{1,q}(\nu, \Omega)\colon D_i u \geqslant D_i w$ a.e. in $\Omega,\ i = 1, \ldots, n\}$, where $w \in W_0^{1,q}(\nu, \Omega)$.

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