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Partial Differential Equations

Nonhomogeneous boundary value problems in anisotropic Sobolev spaces

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Abstract

We study the nonlinear boundary value problem $-\sum_{i=1}^{N}(|u_{x_i}|^{p_i(x)-2}u_{x_i})_{x_i}=\lambda|u|^{q(x)-2}u$ in Ω, $u=0$ on $\partial\Omega$, where $\Omega\subset\mathbb{R}^N$ ($N\geqslant 3$) is a bounded domain with smooth boundary, λ is a positive real number, and the continuous functions p_i and q satisfy $2\leqslant p_i(x)<N$ and $q(x)>1$ for any $x\in\overline{\Omega}$ and any $i\in\{1,\dots,N\}$. By analyzing the growth of the functions p_i and q we prove in this Note several existence results in Sobolev spaces with variable exponents. ***To cite this article: M. Mihăilescu et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Problèmes aux limites non homogènes en espaces de Sobolev anisotropiques. On étudie le problème non linéaire $-\sum_{i=1}^{N}(|u_{x_i}|^{p_i(x)-2}u_{x_i})_{x_i}=\lambda|u|^{q(x)-2}u$ dans Ω, $u=0$ sur $\partial\Omega$, où $\Omega\subset\mathbb{R}^N$ ($N\geqslant 3$) est un domaine borné et régulier, λ est un nombre réel positif et p_i et q sont des fonctions continues telles que $2\leqslant p_i(x)<N$ et $q(x)>1$ pour tout $x\in\overline{\Omega}$ et chaque $i\in\{1,\dots,N\}$. En étudiant la croissance des fonctions p_i et q on obtient dans cette Note plusieurs résultats d'existence dans des espaces de Sobolev aux exposants variables. ***Pour citer cet article : M. Mihăilescu et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

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Soit $\Omega \subset \mathbb{R}^N$ ($N \geqslant 3$) un domaine borné et régulier. On considère le problème non linéaire

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u}{\partial x_i}\right) = \lambda |u|^{q(x)-2} u, & \text{pour } x \in \Omega, \\ u = 0, & \text{pour } x \in \partial\Omega, \end{cases} \tag{1}$$

où $\lambda > 0$ et p_i et q sont des fonctions continues telles que $2 \leqslant p_i(x) < N$ et $q(x) > 1$ pour tout $x \in \overline{\Omega}$ et chaque $i \in \{1, \ldots, N\}$. Pour tout $1 \leqslant j \leqslant N$, soit $p_j^+ := \sup_{x\in\Omega} p_j(x)$, $p_j^- := \inf_{x\in\Omega} p_j(x)$, $P_+^+ := \max\{p_1^+, \ldots, p_N^+\}$, $P_-^+ := \max\{p_1^-, \ldots, p_N^-\}$ et $P_-^- := \min\{p_1^-, \ldots, p_N^-\}$. On suppose que $\sum_{i=1}^N 1/p_i^- > 1$ et on définit $P_-^\star := N/(\sum_{i=1}^N 1/p_i^- - 1)$ et $P_{-,\infty} := \max\{P_-^+, P_-^\star\}$.

Le résultat principal de cette Note est le suivant :

Théorème 0.1.

(a) *On suppose que*

$$P_+^+ < \min_{x\in\overline{\Omega}} q(x) \leqslant \max_{x\in\overline{\Omega}} q(x) < P_-^\star.$$

Alors, pour chaque $\lambda > 0$*, le problème* (1) *admet une solution faible non triviale.*

(b) *On suppose que*

$$1 < \min_{x\in\overline{\Omega}} q(x) < P_-^- \quad \text{et} \quad \max_{x\in\overline{\Omega}} q(x) < P_{-,\infty}.$$

Alors il existe $\lambda^\star > 0$ *tel que pour chaque* $\lambda \in (0, \lambda^\star)$ *le problème* (1) *admet une solution faible non triviale.*

(c) *On suppose que*

$$1 < \min_{x\in\overline{\Omega}} q(x) \leqslant \max_{x\in\overline{\Omega}} q(x) < P_-^-.$$

Alors il existe $\lambda^\star > 0$ *et* $\lambda^{\star\star} > 0$ *tels que pour chaque* $\lambda \in (0, \lambda^\star)$ *et pour chaque* $\lambda > \lambda^{\star\star}$ *le problème* (1) *admet une solution faible non triviale.*

1. The main result

Let $\Omega \subset \mathbb{R}^N$ ($N \geqslant 3$) be a bounded domain with smooth boundary. In this Note we study the nonlinear problem

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u}{\partial x_i}\right) = \lambda |u|^{q(x)-2} u, & \text{for } x \in \Omega, \\ u = 0, & \text{for } x \in \partial\Omega, \end{cases} \tag{2}$$

where $\lambda > 0$ is a real number and the functions p_i and q are continuous on $\overline{\Omega}$ and satisfy $2 \leqslant p_i(x) < N$ and $q(x) > 1$ for any $x \in \overline{\Omega}$ and any $i \in \{1, \ldots, N\}$.

We start with some basic properties of Lebesgue and Sobolev spaces with variable exponents. For any $h \in C_+(\overline{\Omega}) := \{h; h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega}\}$ we define $h^+ := \sup_{x\in\Omega} h(x)$ and $h^- := \inf_{x\in\Omega} h(x)$. For any $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that $\int_\Omega |u(x)|^{p(x)}\, dx < \infty$, where

$$|u|_{p(x)} := \inf\left\{\mu > 0; \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leqslant 1\right\}.$$

An important role is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is defined by $\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\, dx$, for any $u \in L^{p(x)}(\Omega)$. If $u \in L^{p(x)}(\Omega)$ then the following relations hold true

$$|u|_{p(x)} > 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^-} \leqslant \rho_{p(x)}(u) \leqslant |u|_{p(x)}^{p^+}, \tag{3}$$

$$|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^+} \leqslant \rho_{p(x)}(u) \leqslant |u|_{p(x)}^{p^-}. \tag{4}$$

We define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm $\|u\|_{1,p(x)} = \|\nabla u\|_{p(x)}$. The space $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ is a separable and reflexive Banach space. For more details on function spaces with variable exponent we refer to the book by Musielak [8] and the papers by Edmunds et al. [1,2], Kováčik and Rákosník [4], Mihăilescu and Rădulescu [6].

Next, we introduce a natural generalization of the variable exponent Sobolev space $W_0^{1,p(x)}(\Omega)$ that will enable us to study with sufficient accuracy problem (2). Let $\bar{p}(x): \overline{\Omega} \to \mathbb{R}^N$ be the vectorial function $\bar{p}(x) = (p_1(x), \ldots, p_N(x))$. We define the *anisotropic variable exponent Sobolev space* $W_0^{1,\bar{p}(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{\bar{p}(x)} := \sum_{i=1}^N |u_{x_i}|_{p_i(x)}$. In the case where $p_i(x) \in C_+(\overline{\Omega})$ are constant functions denoted by p_i for any $i \in \{1, \ldots, N\}$ the resulting anisotropic Sobolev space is denoted by $W_0^{1,\bar{p}}(\Omega)$, where $\bar{p} = (p_1, \ldots, p_N)$. The theory of such spaces was developed in [3,9,11–13]. It was proved that $W_0^{1,\bar{p}}(\Omega)$ is a reflexive Banach space for any $\bar{p} \in \mathbb{R}^N$ with $p_i > 1$ for all $i \in \{1, \ldots, N\}$. This result can be easily extended to $W_0^{1,\bar{p}(x)}(\Omega)$.

Define $\overline{P}_+ := (p_1^+, \ldots, p_N^+)$, $\overline{P}_- := (p_1^-, \ldots, p_N^-)$ and

$$P_+^+ := \max\{p_1^+, \ldots, p_N^+\}, \quad P_-^+ := \max\{p_1^-, \ldots, p_N^-\}, \quad P_-^- := \min\{p_1^-, \ldots, p_N^-\}.$$

Throughout this Note we assume that

$$\sum_{i=1}^N \frac{1}{p_i^-} > 1. \tag{5}$$

We also define

$$P_-^\star := \frac{N}{\sum_{i=1}^N 1/p_i^- - 1} \quad \text{and} \quad P_{-,\infty} := \max\{P_-^+, P_-^\star\}.$$

The main result of this Note is stated in the following theorem:

Theorem 1.1.

(a) *Assume that the function $q \in C(\overline{\Omega})$ verifies the hypothesis*

$$P_+^+ < \min_{x \in \overline{\Omega}} q(x) \leqslant \max_{x \in \overline{\Omega}} q(x) < P_-^\star. \tag{6}$$

Then for any $\lambda > 0$ problem (2) *has a nontrivial solution in $W_0^{1,\bar{p}(x)}(\Omega)$.*

(b) *Assume that the function $q \in C(\overline{\Omega})$ verifies the hypothesis*

$$1 < \min_{x \in \overline{\Omega}} q(x) < P_-^- \quad \text{and} \quad \max_{x \in \overline{\Omega}} q(x) < P_{-,\infty}. \tag{7}$$

Then there exists $\lambda^\star > 0$ such that for any $\lambda \in (0, \lambda^\star)$ problem (2) *has a nontrivial solution in $W_0^{1,\bar{p}(x)}(\Omega)$.*

(c) *Assume that the function $q \in C(\overline{\Omega})$ verifies the hypothesis*

$$1 < \min_{x \in \overline{\Omega}} q(x) \leqslant \max_{x \in \overline{\Omega}} q(x) < P_-^-. \tag{8}$$

Then there exist $\lambda^\star > 0$ and $\lambda^{\star\star} > 0$ such that for any $\lambda \in (0, \lambda^\star)$ and any $\lambda > \lambda^{\star\star}$ problem (2) *has a nontrivial solution in $W_0^{1,\bar{p}(x)}(\Omega)$.*

2. Proof of Theorem 1.1

The following result extends Theorem 1 in [3]:

Proposition 2.1. *Assume $\Omega \subset \mathbb{R}^N$ ($N \geqslant 3$) is a bounded domain with smooth boundary. Assume relation* (5) *is fulfilled. Assume that $q \in C(\overline{\Omega})$ verifies $1 < q(x) < P_{-,\infty}$, for all $x \in \overline{\Omega}$. Then the embedding $W_0^{1,\vec{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.*

Let E denote the anisotropic variable exponent Sobolev space $W_0^{1,\vec{p}(x)}(\Omega)$.

For any $\lambda > 0$ the energy functional $J_\lambda : E \to \mathbb{R}$ corresponding to problem (2) is defined as

$$J_\lambda(u) = \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} \mathrm{d}x - \lambda \int_\Omega \frac{1}{q(x)} |u|^{q(x)} \, \mathrm{d}x.$$

Proposition 2.1 implies that $J_\lambda \in C^1(E, \mathbb{R})$ and

$$\langle J_\lambda'(u), v \rangle = \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, \mathrm{d}x - \lambda \int_\Omega |u|^{q(x)-2} u v \, \mathrm{d}x,$$

for all $u, v \in E$. Thus, the weak solutions of (2) coincide with the critical points of J_λ.

The following auxiliary results show that J_λ has a mountain-pass geometry:

Lemma 2.2. *There exist $\eta > 0$ and $\alpha > 0$ such that $J_\lambda(u) \geqslant \alpha > 0$ for any $u \in E$ with $\|u\|_{\vec{p}(x)} = \eta$.*

Lemma 2.3. *There exists $e \in E$ with $\|e\|_{\vec{p}(x)} > \eta$ (where η is given in Lemma* 2.2) *such that $J_\lambda(e) < 0$.*

Proof of Theorem 1.1(a). By Lemmas 2.2 and 2.3 and the mountain-pass theorem of Ambrosetti and Rabinowitz we deduce the existence of a sequence $\{u_n\} \subset E$ such that

$$J_\lambda(u_n) \to \bar{c} > 0 \quad \text{and} \quad J_\lambda'(u_n) \to 0 \quad (\text{in } E^\star) \quad \text{as } n \to \infty. \tag{9}$$

We prove that $\{u_n\}$ is bounded in E. Arguing by contradiction, there exists a subsequence (still denoted by $\{u_n\}$) such that $\|u_n\|_{\vec{p}(x)} \to \infty$. Thus, we may assume that for n large enough we have $\|u_n\|_{\vec{p}(x)} > 1$.

For each $i \in \{1, \ldots, N\}$ and any positive integer n we define

$$\alpha_{i,n} = \begin{cases} P_+^+, & \text{if } \left| \frac{\partial u_n}{\partial x_i} \right|_{p_i(x)} < 1, \\ P_-^-, & \text{if } \left| \frac{\partial u_n}{\partial x_i} \right|_{p_i(x)} > 1. \end{cases}$$

So, by relations (9), (3) and (4) we deduce that for n large enough we have

$$\begin{aligned} 1 + \bar{c} + \|u_n\|_{\vec{p}(x)} &\geqslant J_\lambda(u) - \frac{1}{q^-} \langle J_\lambda'(u_n), u_n \rangle \geqslant \left(\frac{1}{P_+^+} - \frac{1}{q^-} \right) \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i(x)} \mathrm{d}x \\ &\geqslant \left(\frac{1}{P_+^+} - \frac{1}{q^-} \right) \sum_{i=1}^{N} \left| \frac{\partial u_n}{\partial x_i} \right|_{p_i(x)}^{\alpha_{i,n}} \mathrm{d}x \\ &\geqslant \left(\frac{1}{P_+^+} - \frac{1}{q^-} \right) \frac{1}{N^{P_{\vec{p}}^-}} \|u_n\|_{\vec{p}(x)}^{P_-^-} - N \left(\frac{1}{P_+^+} - \frac{1}{q^-} \right). \end{aligned} \tag{10}$$

Dividing by $\|u_n\|_{\vec{p}(x)}^{P_-^-}$ in (10) and passing to the limit as $n \to \infty$ we obtain a contradiction. It follows that $\{u_n\}$ is bounded in E. Since E is reflexive, there exists a subsequence, still denoted by $\{u_n\}$, and $u_0 \in E$ such that $\{u_n\}$ converges weakly to u_0 in E. So, by Proposition 2.1, $\{u_n\}$ converges strongly to u_0 in $L^{q(x)}(\Omega)$.

The above considerations and relation (9) imply

$$\lim_{n \to \infty} \sum_{i=1}^{N} \int_\Omega \left(\left| \frac{\partial u_n}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial u_n}{\partial x_i} - \left| \frac{\partial u_0}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial u_0}{\partial x_i} \right) \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u_0}{\partial x_i} \right) \mathrm{d}x = 0. \tag{11}$$

Thus, by (11) and the inequality

$$\left(|\xi|^{r-2}\xi - |\psi|^{r-2}\psi\right)\cdot(\xi-\psi) \geqslant \left(\frac{1}{2}\right)^r |\xi-\psi|^r, \quad \forall r \geqslant 2,\ \forall \xi,\psi \in \mathbb{R}^N,$$

we deduce that $\{u_n\}$ converges strongly to u_0 in E. Then, by (9), $J_\lambda(u_0)=\bar{c}>0$ and $J_\lambda^{'}(u_0)=0$. We conclude that u_0 is a nontrivial weak solution for Eq. (2). The proof of Theorem 1.1(a) is complete.

Proof of Theorem 1.1(b). We start with the following auxiliary result.

Lemma 2.4. *There exists $\lambda^\star>0$ such that for any $\lambda\in(0,\lambda^\star)$ there exist ρ, $a>0$ such that $J_\lambda(u)\geqslant a>0$ for any $u\in E$ with $\|u\|_{\vec{p}(x)}=\rho$.*

A straightforward computation shows that we can choose

$$\lambda^\star := \frac{1}{2P_+^+}\cdot\frac{1}{N^{P_+^+-1}}\cdot\rho^{P_+^+-q^-}\cdot\frac{q^-}{c_1^{q^-}} \quad \text{and} \quad a=\frac{1}{2P_+^+}\cdot\frac{1}{N^{P_+^+-1}}\cdot\rho^{P_+^+}.$$

Let $\lambda^\star>0$ be defined as above and fix $\lambda\in(0,\lambda^\star)$. By Lemma 2.4 it follows that on the boundary of the ball centered at the origin and of radius ρ in E, denoted by $B_\rho(0)$, we have $\inf_{\partial B_\rho(0)} J_\lambda>0$. Standard arguments show that there exists $\phi\in E$, $\phi\geqslant 0$, such that $J_\lambda(t\phi)<0$ for all $t>0$ small enough. Moreover, by (4), for any $u\in B_\rho(0)$ we have

$$J_\lambda(u)\geqslant \frac{1}{P_+^+}\cdot\frac{1}{N^{P_+^+-1}}\cdot\|u\|_{\vec{p}(x)}^{P_+^+}-\frac{\lambda}{q^-}\cdot c_1^{q^-}\cdot\|u\|_{\vec{p}(x)}^{q^-}.$$

It follows that $-\infty<\underline{c}:=\inf_{\overline{B_\rho(0)}} J_\lambda<0$. Fix $0<\epsilon<\inf_{\partial B_\rho(0)} J_\lambda-\inf_{B_\rho(0)} J_\lambda$. Applying Ekeland's variational principle to the functional $J_\lambda:\overline{B_\rho(0)}\to\mathbb{R}$, we find $u_\epsilon\in\overline{B_\rho(0)}$ such that $J_\lambda(u_\epsilon)<\inf_{\overline{B_\rho(0)}} J_\lambda+\epsilon$ and for all $u\neq u_\epsilon$, $J_\lambda(u_\epsilon)<J_\lambda(u)+\epsilon\cdot\|u-u_\epsilon\|_{\vec{p}(x)}$. Since $J_\lambda(u_\epsilon)\leqslant\inf_{\overline{B_\rho(0)}} J_\lambda+\epsilon\leqslant\inf_{B_\rho(0)} J_\lambda+\epsilon<\inf_{\partial B_\rho(0)} J_\lambda$, we deduce that $u_\epsilon\in B_\rho(0)$. Define $I_\lambda:\overline{B_\rho(0)}\to\mathbb{R}$ by $I_\lambda(u)=J_\lambda(u)+\epsilon\cdot\|u-u_\epsilon\|_{\vec{p}(x)}$. Then u_ϵ is a minimum point of I_λ and thus $t^{-1}[I_\lambda(u_\epsilon+t\cdot v)-I_\lambda(u_\epsilon)]\geqslant 0$ for small $t>0$ and any $v\in B_1(0)$. Letting $t\to 0$ it follows that $\langle J_\lambda^{'}(u_\epsilon),v\rangle+\epsilon\cdot\|v\|_{\vec{p}(x)}>0$, hence $\|J_\lambda^{'}(u_\epsilon)\|\leqslant\epsilon$.

We deduce that there exists a sequence $\{w_n\}\subset B_\rho(0)$ such that $J_\lambda(w_n)\to\underline{c}$ and $J_\lambda^{'}(w_n)\to 0$. Moreover, $\{w_n\}$ is bounded in E. Thus, there exists $w\in E$ such that, up to a subsequence, $\{w_n\}$ converges weakly to w in E. Actually, with similar arguments as those used in the end of the proof of Theorem 1.1(a) we can show that $\{w_n\}$ converges strongly to w in E. So, $J_\lambda(w)=\underline{c}<0$ and $J_\lambda^{'}(w)=0$. We conclude that w is a nontrivial weak solution of problem (2). The proof of Theorem 1.1(b) is complete.

Finally, we show that Theorem 1.1(c) holds true. In order to do that, we first point out that by Theorem 1.1(b) it follows that there exists $\lambda^\star>0$ such that for any $\lambda\in(0,\lambda^\star)$ problem (2) has a nontrivial weak solution. In order to show that there exists $\lambda^{\star\star}>0$ such that for any $\lambda>\lambda^{\star\star}$ problem (2) has a nontrivial weak solution, we prove that J_λ possesses a nontrivial global minimum point in E. The same arguments as in the proof of Lemma 3.4 in [5] can be used to show that J_λ is weakly lower semicontinuous on E. Moreover, J_λ is coercive on E. So, by Theorem 1.2 in [10], there exists a global minimizer $u_\lambda\in E$ of J_λ and, thus, a weak solution of problem (2). We show that u_λ is not trivial for λ large enough. Indeed, letting $t_0>1$ be a fixed real and Ω_1 be an open subset of Ω with $|\Omega_1|>0$, we deduce that there exists $v_0\in C_0^\infty(\Omega)\subset E$ such that $v_0(x)=t_0$ for any $x\in\overline{\Omega}_1$ and $0\leqslant v_0(x)\leqslant t_0$ in $\Omega\setminus\Omega_1$. We have

$$J_\lambda(v_0)=\sum_{i=1}^N\int_\Omega \frac{1}{p_i(x)}\left|\frac{\partial v_0}{\partial x_i}\right|^{p_i(x)}\mathrm{d}x-\lambda\int_\Omega\frac{1}{q(x)}|v_0|^{q(x)}\,\mathrm{d}x\leqslant L-\frac{\lambda}{q^+}\int_{\Omega_1}|v_0|^{q(x)}\,\mathrm{d}x\leqslant L-\frac{\lambda}{q^+}\cdot t_0^{q^-}\cdot|\Omega_1|$$

where L is a positive constant. Thus, there exists $\lambda^{\star\star}>0$ such that $J_\lambda(u_0)<0$ for any $\lambda\in[\lambda^{\star\star},\infty)$. It follows that $J_\lambda(u_\lambda)<0$ for any $\lambda\geqslant\lambda^{\star\star}$ and thus u_λ is a nontrivial weak solution of problem (2) for λ large enough. The proof of Theorem 1.1(c) is complete.

We refer to [7] for complete proofs and additional results.

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References

[1] D.E. Edmunds, J. Lang, A. Nekvinda, On $L^{p(x)}$ norms, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 455 (1999) 219–225.
[2] D.E. Edmunds, J. Rákosník, Sobolev embedding with variable exponent, Studia Math. 143 (2000) 267–293.
[3] I. Fragalà, F. Gazzola, B. Kawohl, Existence and nonexistence results for anisotropic quasilinear equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (2004) 715–734.
[4] O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{1,p(x)}$, Czechoslovak Math. J. 41 (1991) 592–618.
[5] M. Mihăilescu, V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 462 (2006) 2625–2641.
[6] M. Mihăilescu, V. Rădulescu, On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, Proc. Amer. Math. Soc. 135 (2007) 2929–2937.
[7] M. Mihăilescu, P. Pucci, V. Rădulescu, Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent, J. Math. Anal. Appl., in press, doi:10.1016/j.jmaa.2007.09.015.
[8] J. Musielak, Orlicz Spaces and Modular Spaces, Lecture Notes in Math., vol. 1034, Springer, Berlin, 1983.
[9] S.M. Nikol'skii, On imbedding, continuation and approximation theorems for differentiable functions of several variables, Russian Math. Surveys 16 (1961) 55–104.
[10] M. Struwe, Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Springer, Heidelberg, 1996.
[11] M. Troisi, Teoremi di inclusione per spazi di Sobolev non isotropi, Ricerche Mat. 18 (1969) 3–24.
[12] N. Trudinger, On embeddings into Orlicz spaces and some applications, J. Math. Mech. 17 (1967) 473–483.
[13] L. Ven'-tuan, On embedding theorems for spaces of functions with partial derivatives of various degree of summability, Vestnik Leningrad. Univ. 16 (1961) 23–37.