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Numerical Analysis

A posteriori error estimations of a SUPG method for anisotropic diffusion–convection–reaction problems

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Abstract

This Note presents an a posteriori residual error estimator for diffusion–convection–reaction problems approximated by a SUPG scheme on isotropic or anisotropic meshes in $\mathbb{R}^d$, $d = 2$ or 3. This estimator is based on the jump of the flux and the interior residual of the approximated solution. It is constructed to work on anisotropic meshes which account for the eventual anisotropic behavior of the solution. The equivalence between the energy norm of the error and the estimator is proved. ***To cite this article: T. Apel, S. Nicaise, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Estimations d'erreur a posteriori d'une méthode SUPG pour des problèmes de diffusion–convection–réaction anisotropes. Cette Note présente un estimateur d'erreur a posteriori du type résiduel pour des problèmes de diffusion–convection–réaction approché par un schéma SUPG sur des triangulations isotropes ou non de $\mathbb{R}^d$, $d = 2$ or 3. Cet estimateur est basé sur les sauts des flux et les résidus intérieurs de la solution approchée. Il est construit pour fonctionner sur des maillages anisotropes qui tiennent en compte l'anisotropie éventuelle de la solution. L'équivalence entre la norme de l'erreur et l'estimateur est démontrée. ***Pour citer cet article : T. Apel, S. Nicaise, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

Dans cette Note nous considèrons le problème (1) de diffusion–convection–réaction singulièrement perturbé avec un attention particulière au cas où la matrice de diffusion est anisotrope. Ce problème est posé dans un domaine borné $\Omega \subset \mathbb{R}^d$, $d = 2$ ou 3, à bord Γ polygonal ($d = 2$) ou polyhèdral ($d = 3$), divisé en deux parties Γ_D et Γ_N ; les coefficients $A \in L^\infty(\Omega)^{d\times d}$, $\mathbf{b} \in W^{1,\infty}(\Omega)^d$ et $c \in L^\infty(\Omega)$ vérifiant les hypothèses décrites ci-dessous.

Nous sommes particulièrement interessé au cas où A devient petit dans une direction, par exemple le cas $A = \operatorname{diag}(\varepsilon, 1)$ si $d = 2$ lorsque $\varepsilon > 0$ devient petit. Dans ce cas, le problème est singulièrement perturbé et la solution peut présenter des couches limites intérieures ou de bord, là ou la solution du problème limite (correspondant à $\varepsilon = 0$)

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est non régulière ou ne satisfait pas la condition au bord. Citons [10] où une analyse d'erreur a priori en dimension deux a montré que l'utilisation de maillages anisotropes permet d'obtenir une convergence uniforme par rapport au paramètre ε.

Il existe une vaste littérature sur les estimations d'erreur a posteriori. Pour des problèmes singulièrement perturbés avec convection on peut citer [2,6,9,12,14], où des maillages anisotropes ont été considérés dans [9,12] seulement. Le seul cas où une matrice de diffusion anisotrope a été considéré est dans le travail récent [5].

Dans ce travail nous combinons tous ces ingrédients. Nous utilisons des maillages anisotropes et proposons un estimateur d'erreur du type résiduel. Nous démontrons son efficacité et sa fiabilité, où la dépendance en fonction de ε est spécifiée. La borne inférieure dépend essentiellement du nombre local de Peclet $Pe_T := h_{\min,A,T}\|A^{-1/2}\mathbf{b}\|_{\infty,T}$, dès lors l'efficacité est atteinte si $Pe_T \leqslant c$, ce qui est toujours satisfait lorsqu'il n'y a pas de convection. La fiabilité est basée sur l'introduction d'un *mesure d'alignement* comme dans Kunert [7,8]. Cette quantité reste bornée si le maillage est bien adapté au problème.

Signalons qu'à notre connaissance il n'y a aucune approche connue qui conduise à des estimées efficaces et fiables sur des maillages anisotropes sans aucune condition sur ces maillages. Les résultats classiques comme présentés dans [1,13] sont obtenus pour des maillages isotropes exclusivement. L'approche plus récente de [11] n'a pas encore été investigée pour des maillages anisotropes et de plus deux estimateurs différents sont utilisés pour obtenir la borne supérieure et la borne inférieure. Finalement mentionnons l'approche de Picasso [12] où l'auteur considère des maillages anisotropes et prouve la fiabilité de son estimateur qui en fait dépend de $\nabla(u-u_h)$ et où, en pratique, ∇u est remplacé par un gradient reconstitué $\nabla^R u$. Notons que nous pouvons contrôler notre mesure d'alignement de la même manière.

Par rapport à [5], où une estimation d'erreur a posteriori a été investigée pour des maillages isotropes pour un problème avec une matrice de diffusion anisotrope mais sans convection, notre estimateur permet de prouver une borne inférieure optimale. Le facteur $\varepsilon^{-1/2}$ dans la borne supérieure de [5] est conservée dans notre analyse, puisque la mesure d'alignement est de cet ordre pour le cas de maillages isotropes. Néanmoins les tests montrent que la mesure d'alignement est bornée lorsque les maillages anisotropes sont rafinées de manière adéquate. Dans ce sens, notre analyse est plus fine.

Notre problème est discrétisé en utilisant une h-version de la méthode de Petrov–Galerkin amont (SUPG). Sans le terme de stabilisation, cette méthode se réduit à une méthode de Galerkin standard qui produit des oscillations non-physiques, néanmoins notre estimateur peut s'appliquer dans ce cas.

La formulation variationelle du problème (1) consiste à chercher $u \in V$ solution de (2) qui a une solution unique grâce au lemme de Lax–Milgram, puisque les hypothèses faites sur les coefficients garantissent que B est continue et coercive.

Pour approcher le problème (2) on fixe une famille de triangulations $\{T_h\}_{h>0}$ de Ω qui satisfait les conditions de conformité usuelle, cf. [4, Chapter 2]. Nous supposons que tous les éléments de T_h sont des triangles si $d=2$ et des tétrahèdres si $d=3$. Nous supposons de plus que T_h est consistant avec les discontinuités de A, i.e., A est constant sur chaque élément $T \in T_h$. Pour $T \in T_h$ on désigne par h_T le diamètre de T, et $h = \max_{T\in T_h} h_T$.

On définit alors V_h comme le sous-espace de V constitué de polynômes de degré un par morceaux. Le problème (2) est ainsi approché par le schéma (3), dit de Petrov Galerkin amont (SUPG).

L'estimateur η étant défini à la section 3, nous montrons qu'il est efficace et fiable. Autrement dit nous montrons, en posant $\kappa_T = \max_{T'\cap T\neq\emptyset}\{\max\{1, c_0^{-1}\|c\|_{\infty,T'}\} + Pe_{T'}\}$, que

$$\begin{aligned}
&|\!|\!|u-u_h|\!|\!| \lesssim m_1(u-u_h, A, T_h)\,\eta,\\
&\eta_T \lesssim \kappa_T\Bigg(\sum_{T'\cap T\neq\emptyset}\int_{T'}\big|A\nabla(u-u_h)\big|^2 + c_0|u-u_h|^2\Bigg)^{1/2}.
\end{aligned}$$

1. Introduction

This Note is devoted to the singularly perturbed diffusion–convection–reaction problem with special focus on anisotropic diffusion: for $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$, let u be the solution of

$$\begin{cases} Lu := -\operatorname{div}(A\nabla u) + \mathbf{b}\cdot\nabla u + cu = f \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \Gamma_D, \qquad A\nabla u\cdot n = g \quad \text{on } \Gamma_N, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, is a bounded domain with a polygonal ($d = 2$) or polyhedral ($d = 3$) boundary Γ, divided into two parts Γ_D and Γ_N; $A \in L^\infty(\Omega)^{d\times d}$ is symmetric, piecewise constant, and uniformly positive definite ($\exists\alpha_0 > 0$: $A(x)\xi\cdot\xi \geqslant \alpha_0$, $\forall\xi \in \mathbb{R}^d$, for a.a. $x \in \Omega$); $\mathbf{b} \in W^{1,\infty}(\Omega)^d$ and $c \in L^\infty(\Omega)$ satisfy the next assumptions:

$$\exists c_0 > 0\colon 2c - \operatorname{div}\mathbf{b} \geqslant 2c_0 \quad \text{in } \Omega, \quad \text{and} \quad \mathbf{b}\cdot n \geqslant 0 \quad \text{on } \Gamma_N.$$

Our analysis below can also be made in the case $c_0 = 0$; in that case we require that $c \equiv 0$ and $\Gamma_D \neq \emptyset$.

We are particularly interested in the case when A becomes small in some direction, for instance the case $A = \operatorname{diag}(\varepsilon, 1)$ if $d = 2$ when $\varepsilon > 0$ becomes small. In this case, the problem is singularly perturbed and the solution may generate sharp boundary or interior layers, where the solution of the limit problem (corresponding to $\varepsilon = 0$) is not smooth or does not satisfy the boundary condition. It is shown in two dimensions in [10] that anisotropic meshes must be used in order to achieve convergence uniform in the perturbation parameter ε.

There is a vast amount of literature on a posteriori error estimation. For singularly perturbed problems with convection we cite [2,6,9,12,14], where anisotropic finite element meshes were considered in [9,12] only. An anisotropic diffusion tensor is considered only in [5].

In this Note we combine all those ingredients. We discretize with anisotropic meshes and derive a residual type error estimator. We prove the reliability and efficiency of this error estimator where the dependence on ε is traced. The lower bound mainly depends on the local mesh Peclet number

$$Pe_T := h_{\min,A,T}\|A^{-1/2}\mathbf{b}\|_{\infty,T},$$

therefore the efficiency is achieved if $Pe_T \leqslant c$ which is always satisfied in the absence of convection. The reliability is based on the introduction of a *matching function* as it was done by Kunert [7,8]. This quantity is of the order one if the mesh is well adapted to the problem.

Let us mention that, to our knowledge, no approach is known that leads to two-sided estimates on anisotropic meshes without any assumption on the mesh. The classical results as summarized in [1,13] are obtained for isotropic meshes only. The more recent approach in [11] is not yet analyzed for anisotropic meshes and two different error estimators are used for the upper and lower bounds. Let us finally mention the approach by Picasso [12] who considers anisotropic meshes and proves reliability for an estimator that depends on $\nabla(u - u_h)$ where ∇u is replaced in practice by a recovered gradient $\nabla^R u$. We note that we can control in the same way the matching function.

In comparison with the paper [5], where a posteriori error estimation is investigated for an isotropic discretization of a problem with anisotropic diffusion but without convection, our residual error estimator allows to prove an optimal lower bound. The factor $\varepsilon^{-1/2}$ in the upper bound in [5] is retained in our analysis, since the matching function is of the same order in the isotropic case. Our experiments show, however, that the matching function is of order one on adequately refined anisotropic meshes. In this sense, our analysis is sharper.

For the discretization we use the h-version of the streamline upwind Petrov–Galerkin method (SUPG). Without the stabilization term, the method reduces to a standard Galerkin method and produces non-physical oscillations; nevertheless our error estimator works as well.

Now we define $V = \{v \in H^1(\Omega)\colon u = 0 \text{ on } \Gamma_D\}$ equipped with the norm

$$|\!|\!|u|\!|\!|^2 := \int_\Omega \left(|A\nabla u|^2 + c_0|u|^2\right)$$

and introduce the forms

$$B(u,v) = \int_\Omega (A\nabla u\cdot\nabla v + \mathbf{b}\cdot\nabla u v + cuv)\,\mathrm{d}x, \qquad F(v) = \int_\Omega f v\,\mathrm{d}x + \int_{\Gamma_N} g v\,\mathrm{d}\Gamma(x) \quad \forall u, v \in V.$$

With this notation, the variational formulation of problem (1) reads: Find $u \in V$ solution of

$$B(u,v) = F(v) \quad \forall v \in V. \tag{2}$$

The above assumptions guarantee that B is continuous and coercive (i.e. $B(v,v) \geqslant |\!|\!|v|\!|\!|^2$, $\forall v \in V$). By the Lax–Milgram lemma, problem (2) has a unique solution $u \in V$.

To approximate problem (2) we fix a family $\{T_h\}_{h>0}$ of meshes of Ω that satisfies the usual conformity conditions, cf. [4, Chapter 2]. We assume that all elements of T_h are triangles if $d = 2$ and tetrahedra if $d = 3$. We further suppose that T_h is conforming with the discontinuities of A, i. e., A is constant on each element $T \in T_h$. For $T \in T_h$ we denote by h_T the diameter of T, and $h = \max_{T\in T_h} h_T$.

Let V_h be the subspace of V defined by $V_h = \{v_h \in V\colon\ v_{h|T} \in \mathbb{P}^1(T) \forall T \in T_h\}$. Set

$$B_h(u_h, v_h) = B(u_h, v_h) + \sum_{K\in T_h} \delta_T (Lu_h, \mathbf{b}\cdot\nabla v_h)_T, \qquad F_h(v_h) = F(v_h) + \sum_{K\in T_h} \delta_T (f, \mathbf{b}\cdot\nabla v_h)_T.$$

The parameters $\delta_T \geqslant 0$ satisfy similar assumptions as in [9] where only isotropic diffusion was investigated,

$$\delta_T \leqslant \min\Big\{\mu^{-2} h^2_{\min,T}\Big(\max_{x\in T}\|A(x)^{1/2}\|_{2\to 2}\Big)^{-2}, c_0\Big(\max_{x\in T} c(x)^2\Big)^{-1}, h_{\min,A,T}\|A^{-1/2}\mathbf{b}\|^{-1}_{\infty,T}\Big\}, \quad \forall T \in T_h,$$

where μ is the constant in the inverse inequality $\|\operatorname{div}(\nabla v_h)\|_T \leqslant \mu h^{-1}_{\min,T}\|\nabla v_h\|_T$. Problem (2) is now approximated by a SUPG scheme: Find $u_h \in V_h$ solution of

$$B_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h. \tag{3}$$

2. Some analytical tools

Let us define E_h as the set of edges ($d = 2$) or faces ($d = 3$) of the triangulation and define by $E_h^{\mathrm{int}} = \{E \in E_h\colon E \subset \Omega\}$ the set of interior edges/faces of T_h, while $E_h^{\mathrm{ext}} = E_h \setminus E_h^{\mathrm{int}}$ is the set of boundary edges/faces of T_h. For each edge/face E we fix one of the two normal vectors and denote it by n_E.

As explained in the Introduction, anisotropic discretizations can be very advantageous or, in certain situations, even mandatory. More information and arguments concerning anisotropy can be found [3,7]. The key idea is to transform any $T \in T_h$ by the matrix $A^{-1/2}$. More precisely, we set $A_T = A(g_T)$, where g_T is the center of gravity of T. Then we transform T into T_A by the affine transformation $T \to T_A : x \to A_T^{-1/2}(x - g_T) + g_T$. For this element T_A we use its anisotropic quantity $h_{\min,T_A}$ and matrix C_{T_A} introduced by Kunert in [7] (see also [9]). For use later, we denote

$$h_{\min,A,T} = \min_{T'\subset\omega_T} h_{\min,T'_A}, \quad C_{A,T} = A_T^{1/2} C_{T_A}, \quad \alpha_T = \min\{c_0^{-1/2}, h_{\min,A,T}\}.$$

We finally require that $|T| \sim |T'|$ if $T \cap T' \neq \emptyset$, and that the number of elements containing a vertex x is bounded uniformly.

In order to obtain an accurate discrete solution u_h, it is obviously helpful to align the elements of the mesh according to the anisotropy of the solution. It turns out that this intuitive alignment is also necessary to prove sharp upper error bounds. In particular the proof employs specific interpolation error estimates. These interpolation estimates hold for isotropic meshes, but do not hold for general anisotropic meshes; instead the mesh has to have the aforementioned anisotropic alignment with the function to be interpolated. In order to quantify this alignment, we introduce for $v \in H^1(\Omega)$ the alignment measure:

$$m_1(v, A, T_h) := \Big(\sum_{T\in T_h} h^{-2}_{\min,A,T}\ |C^\top_{A,T}\nabla v\|^2_T\Big)^{1/2} \Big/ \|A^{1/2}\nabla v\| \tag{4}$$

which was originally introduced in [8] for the identity matrix and that we extend here to any matrix A. For arbitrary isotropic meshes one obtains that $m_1(v, Id, T_h) \sim 1$. The same is achieved for anisotropic meshes T_h that are aligned with the anisotropic function v.

Now we recall the so-called Clément interpolation operator that maps a function from V into V_h. $\mathrm{I_{Cl}}v$ is the unique element in V_h defined by $\mathrm{I_{Cl}}v(x) = (\int_{\omega_x} v)/|\omega_x|$, for all nodes x of the triangulation included into Ω and Γ_N, where ω_x is the set of $T \in T_h$ having x as vertex. Using some scaling arguments one can prove the following global interpolation error bound:

Lemma 2.1. *For each edge/face E, let us set $\beta_E = \max_{T\subset\omega_E}(h_{E,T} h^{-1}_{\min,A,T})$, where $h_{E,T} = |T|/|E|$ is the height of E. Let $v \in V$, then*

$$|\!|\!|\mathrm{I_{Cl}}v|\!|\!|^2 + \sum_{T\in T_h}\Big(\alpha_T^{-2}\|v - \mathrm{I_{Cl}}v\|^2_T + \alpha_T^{-1}\sum_{E\in\partial T\setminus\Gamma_D}\beta_E\|v - \mathrm{I_{Cl}}v\|^2_E\Big) \lesssim m_1(v, A, T_h)^2|\!|\!|v|\!|\!|^2. \tag{5}$$

3. Error estimators

The exact element residual is defined by $R_T := f - Au_h$ on T. Similarly the exact edge/face residual is $R_E = [\![A\nabla u_h \cdot n_E]\!]_E$ on $E \in E_h^{\text{int}}$ (where for all $y \in E$, $[\![v(y)]\!]_E := \lim_{\alpha\to+0} v(y+\alpha n_E) - v(y-\alpha n_E)$), $R_E = g - A\nabla u_h \cdot n$ on $E \in E_h^{\text{ext}} \cap \Gamma_N$, and $R_E = 0$ on $E \in E_h^{\text{ext}} \cap \Gamma_D$. As usual, these exact residuals are replaced by some finite-dimensional approximation r_T and r_E called approximate residuals. For shortness we do not take into account this replacement here. The local and global residual error estimators are defined by

$$\eta_T^2 := \alpha_T^2 \|R_T\|_T^2 + \alpha_T \sum_{E\in\partial T\setminus\Gamma_D} \beta_E^{-1} \|R_E\|_E^2, \qquad \eta^2 := \sum_{K\in T_h} \eta_T^2.$$

Theorem 3.1. *Let u be a solution of* (2) *and u_h a solution of* (3). *Then the error is bounded as follows*:

$$|\!|\!| u - u_h |\!|\!| \lesssim m_1(u - u_h, A, T_h)\eta.$$

Proof. For shortness we write $v = u - u_h$ and by the coerciveness of B we have

$$|\!|\!| v |\!|\!|^2 \leqslant B(v, v) \leqslant B(v, v - \mathrm{I}_{\mathrm{Cl}} v) + B(v, \mathrm{I}_{\mathrm{Cl}} v). \tag{6}$$

For the first term, elementwise integration by parts yields

$$B(v, v - \mathrm{I}_{\mathrm{Cl}} v) = \sum_{K\in T_h} (R_T, v - \mathrm{I}_{\mathrm{Cl}} v)_T + \sum_{E\in E_h} (R_E, v - \mathrm{I}_{\mathrm{Cl}} v)_E.$$

By continuous and discrete Cauchy–Schwarz's inequality and the use of the estimate (5), we arrive at

$$B(v, v - \mathrm{I}_{\mathrm{Cl}} v) \lesssim m_1(v, A, T_h)\eta |\!|\!| v |\!|\!|.$$

For the second term we use scaling arguments, the assumption on δ_T and the estimate (5) to show that

$$B(v, \mathrm{I}_{\mathrm{Cl}} v) \lesssim m_1(v, A, T_h)\eta |\!|\!| v |\!|\!|.$$

The lower bound are proved using standard elementwise integration by parts and some inverse estimates. □

Theorem 3.2. *For all elements T, if $\kappa_T = \max_{T'\cap T\neq\emptyset}\{\max\{1, c_0^{-1}\|c\|_{\infty,T'}\} + Pe_{T'}\}$, then*

$$\eta_T \lesssim \kappa_T \left(\sum_{T'\cap T\neq\emptyset} \int_{T'} \left|A\nabla(u - u_h)\right|^2 + c_0|u - u_h|^2 \right)^{1/2}.$$

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