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Differential Geometry/Differential Topology

The topology of the space of symplectic balls in $S^2 \times S^2$

Sílvia Anjos [a,1], François Lalonde [b,2]

[a] Centro de Análise Matemática, Geometria e Sistemas Dinâmicos, Departamento de Matemática, Instituto Superior Técnico, Lisboa, Portugal
[b] Département de mathématiques et de statistique, Université de Montréal, C.P. 6128 succ. Centre–Ville, Montréal QC H3C 3J7, Canada

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Abstract

In this Note we compute the full homotopy type of the space of symplectic embeddings of the standard ball $B^4(c) \subset \mathbb{R}^4$ with capacity $c = \pi r^2$ into the 4-dimensional rational symplectic manifold $M_\mu = (S^2 \times S^2, \mu\omega_0 \oplus \omega_0)$ where μ belongs to the interval $(1, 2]$ and c is above the critical value $\mu - 1$. ***To cite this article: S. Anjos, F. Lalonde, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

La tolologie de l'espace des boules symplectiques dans $S^2 \times S^2$. Dans cette Note, nous calculons le type d'homotopie complet de l'espace des plongements symplectiques de la boule standard $B^4(c) \subset \mathbb{R}^4$ de capacité $c = \pi r^2$ dans la 4-variété rationnelle $M_\mu = (S^2 \times S^2, \mu\omega_0 \oplus \omega_0)$ où μ appartient à l'intervalle $(1, 2]$ et c est plus grand que la valeur critique $\mu - 1$. ***Pour citer cet article : S. Anjos, F. Lalonde, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

This Note is the follow-up and conclusion of the paper [5]. All proofs can be found in [2]. Consider the rational symplectic manifold $M_\mu = (S^2 \times S^2, \mu\omega_0 \oplus \omega_0)$ where ω_0 is the area form on the sphere with total area 1 and where μ belongs to the interval $(1, 2]$. Let $B^4(c) \subset \mathbb{R}^4$ be the closed standard ball of radius r and capacity $c = \pi r^2$ equipped with the restriction of the symplectic structure $\omega_{st} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ of $\mathbb{R}^4$. Let $\mathrm{Emb}_\omega(c, \mu)$ be the space, endowed with the C^∞-topology, of all symplectic embeddings of $B^4(c)$ in M_μ. Finally, let $\Im\mathrm{Emb}_\omega(c, \mu)$ be the space of subsets of M that are images of maps belonging to $\mathrm{Emb}_\omega(c, \mu)$ defined as the topological quotient

$$\mathrm{Symp}\big(B^4(c)\big) \hookrightarrow \mathrm{Emb}_\omega(c, \mu) \longrightarrow \Im\mathrm{Emb}_\omega(c, \mu)$$

where $\mathrm{Symp}(B^4(c))$ is the group, endowed with the C^∞-topology, of symplectic diffeomorphisms of the closed ball, with no restrictions on the behavior on the boundary (thus each such map extends to a symplectic diffeomorphism of a neighborhood of $B^4(c)$ that sends $B^4(c)$ to itself). We may view $\Im\mathrm{Emb}_\omega(c, \mu)$ as the space of all unparametrized balls of capacity c of M.

E-mail addresses: sanjos@math.ist.utl.pt (S. Anjos), lalonde@dms.umontreal.ca (F. Lalonde).
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One knows from [5] that $\mathrm{Emb}_\omega(c,\mu)$ retracts to the space of symplectic frames of $M = S^2 \times S^2$ for all values $c < \mu - 1$ (hence the space $\Im\mathrm{Emb}_\omega(c,\mu)$ retracts to $S^2 \times S^2$ in this range of c's). Recall that the Non-squeezing Theorem implies that this space is empty for $c \geqslant 1$. It is shown moreover in [5,6] that, if $\ell < \mu \leqslant \ell + 1$ where $\ell \in \mathbb{N}$, the homotopy type of $\Im\mathrm{Emb}_\omega(c,\mu)$ changes only when μ passes an integer or c crosses the critical value $\mu - \ell$ (and that it does change at those values). In this note we study the full homotopy type of $\Im\mathrm{Emb}_\omega(c,\mu)$ for values $\mu - 1 \leqslant c < 1$ when $1 < \mu \leqslant 2$ and show, in particular, that it does not have the homotopy type of a finite dimensional CW-complex. The generalization to all values of μ, that is to say to the cases in which μ lies in the interval $(n, n+1]$ with $n \in \mathbb{N}$, is treated in the forthcoming paper [3]. It requires a deeper study of the topology of the group of symplectic diffeomorphisms of M_μ and its classifying space. In that paper we also study the topology of the space of all symplectic embeddings of $B^4(c)$ in $(\mathbb{CP}^2\#\overline{\mathbb{CP}^2}, \omega_\mu)$ where the symplectic area of the exceptional divisor is $\mu > 0$ and the area of a projective line is $\mu + 1$.

Now we briefly recall the background needed from [5]. Denote by ι_c, $c < 1$, the standard symplectic embedding of $B^4(c)$ in M_μ. It is defined as the composition $B^4(c) \hookrightarrow D^2(\mu - \epsilon) \times D^2(1 - \epsilon) \hookrightarrow S^2(\mu) \times S^2(1) = M_\mu$ where the parameters between parentheses represent the areas. It is shown in [4] and [5] that there is a Serre fibration

$$\mathrm{Symp}(\widetilde{M}_{\mu,c}) \hookrightarrow \mathrm{Symp}(M_\mu) \longrightarrow \Im\mathrm{Emb}_\omega(c,\mu) \tag{1}$$

where the space in the middle is the group of all symplectic diffeomorphisms of M_μ, $\widetilde{M}_{\mu,c}$ is the blow-up of M_μ at the ball ι_c and $\mathrm{Symp}(\widetilde{M}_{\mu,c})$ is the group of its symplectomorphisms.

The full homotopy type of the middle group has been computed by Anjos–Granja in [1]. To explain their result, recall first that the Hirzebruch surface W_i is given by $W_i = \{([z_0, z_1], [w_0, w_1, w_2]) \in \mathbb{CP}^1 \times \mathbb{CP}^2 | z_0^i w_1 = z_1^i w_0\}$ and it is well known that the restriction of the projection $\pi_1 : \mathbb{CP}^1 \times \mathbb{CP}^2 \to \mathbb{CP}^1$ to W_i endows W_i with the structure of a Kähler $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$ which is topologically $S^2 \times S^2$ if i is even. Moreover the group $\mathrm{SO}(3) \times \mathrm{SO}(3)$ can be considered as a subgroup of $\mathrm{Symp}(M_\mu)$ by letting each factor acts on the corresponding factor of M_μ and the group $\mathrm{SO}(3) \times S^1$ is also a subgroup of $\mathrm{Symp}(M_\mu)$ by carrying through the symplectomorphism $W_2 \to M_\mu$ the Kahler isometry group $\mathrm{SO}(3) \times S^1$ of the Hirzebruch surface W_2. Here the S^1-factor is the rotation in the fibers of W_2 – it is therefore the 'rotation' in $M_\mu = S^2 \times S^2$ of the fibers of the projection onto the first factor, whereas $\mathrm{SO}(3)$ is a lift to $W_2 \to \mathbb{CP}^1 = S^2$ of the group $\mathrm{SO}(3)$ on the base. This gives two subgroups of $\mathrm{Symp}(M_\mu)$ and it turns out, by a result of Abreu, that the first factor of $\mathrm{SO}(3) \times S^1$ can be identified with the diagonal in $\mathrm{SO}(3) \times \mathrm{SO}(3)$. Hence we have the following diagram

$$\begin{array}{ccc} \mathrm{SO}(3) & \xrightarrow{\ \Delta\ } & \mathrm{SO}(3) \times \mathrm{SO}(3) \\ \Big\downarrow{\scriptstyle i} & & \\ \mathrm{SO}(3) \times S^1 & & \end{array} \tag{2}$$

where Δ denotes the inclusion of the diagonal and i is the inclusion of the first factor.

This induces a map from the pushout $P = (\mathrm{SO}(3) \times \mathrm{SO}(3)) \coprod_{\mathrm{SO}(3)} (\mathrm{SO}(3) \times S^1)$ (or amalgamated product) to $\mathrm{Symp}(M_\mu)$ by the universal property of pushouts. Recall that the pushout, in the category of topological groups, is characterized as the initial object in the category of topological groups admitting compatible homomorphisms from diagram (2). In [1], Anjos–Granja proved that if $1 < \mu \leqslant 2$ this H-map is a weak homotopy equivalence. In this computation the tools of algebraic topology became very useful.

The computation of the homotopy type of $\mathrm{Symp}(\widetilde{M}_{\mu,c})$ as a topological group relies on results obtained in [5,6] regarding the topology of this group together with techniques previously used in [1]. Using the same notations as in [5], let the 2-torus T_i^2 be the group of Kähler isometries of the blow up $\widetilde{W}_{i,c}$ of the Hirzbruch surface W_i at a standard ball of capacity $c < 1$ centered at a point on the zero section of W_i [5, Prop. 4.4 and Prop. 4.5]. Each torus T_i^2 gives rise to an abelian subgroup of $\mathrm{Symp}(\widetilde{M}_{\mu,c})$ that we will denote by $\widetilde{T}_i^2$. When $\mu \in (1,2]$ and $c \geqslant \mu - 1$, only the tori $\widetilde{T}_0^2$ and $\widetilde{T}_1^2$ exist. It turns out that the first torus is the product $S^1 \times S^1$, that can be considered as subgroup of the group $\mathrm{SO}(3) \times \mathrm{SO}(3)$ of diagram (2) (when $\mathrm{Symp}(\widetilde{M}_{\mu,c})$ is thought of as the subgroup of $\mathrm{Symp}(M_\mu)$ that preserves – not necessarily pointwise – the ball of capacity c – see [5]). The second torus may be viewed as the subgroup $S^1 \times S^1 \subset \mathrm{SO}(3) \times S^1$ where the second factors are identified and where the first S^1-factor is included in $\mathrm{SO}(3)$ as the subgroup of the Kähler isometries of W_2 that preserves a point on the section at infinity of W_2. More precisely, we show that the group $\mathrm{Symp}(\widetilde{M}_{\mu,c})$ has the homotopy type of a pushout.

Theorem 1. *If* $0 < \mu - 1 \leqslant c < 1$, *then the H-map*

$$\widetilde{P} = \widetilde{T_0^2} \coprod_{S^1} \widetilde{T_1^2} \to \mathrm{Symp}(\widetilde{M}_{\mu,c})$$

is a weak homotopy equivalence of topological groups.

We can now state the main result:

Theorem 2. *If* $0 < \mu - 1 \leqslant c < 1$, *the topological space* $\Im\mathrm{Emb}_\omega(c,\mu)$ *is weakly homotopy equivalent to the quotient* $P/\widetilde{P}$ *of the two pushouts*; *moreover, this quotient space is the total space of a non-trivial fibration*

$$\Omega\Sigma^2\mathrm{SO}(3)/\Omega S^3 \xrightarrow{\bar{\iota}} P/\widetilde{P} \xrightarrow{\bar{\pi}} S^2 \times S^2 \tag{3}$$

where the inclusion of the group ΩS^3 *in* $\Omega\Sigma^2\mathrm{SO}(3)$ *is induced by the map* $S^3 \to \Sigma^2\mathrm{SO}(3)$ *that corresponds to the generator of the fundamental group of* $\mathrm{SO}(3)$. *This fibration has a continuous section and splits homotopically, i.e.*

$$\pi_k\big(\Im\mathrm{Emb}_\omega(c,\mu)\big) \simeq \pi_k\big(\Omega\Sigma^2\mathrm{SO}(3)/\Omega S^3\big) \oplus \pi_k(S^2 \times S^2).$$

Sketch of the proof. The topological group $G = \mathrm{SO}(3) \times \mathrm{SO}(3) \times S^1$ sits in the obvious square diagram obtained from diagram (2) with continuous homomorphisms $g_1 : \mathrm{SO}(3) \times \mathrm{SO}(3) \to G$ and $g_2 : \mathrm{SO}(3) \times S^1 \to G$ given by $g_1(a,b) = (a,b,1)$ and $g_2(c,d) = (c,c,d)$. It follows from the universal property of the pushout that there is a canonical continuous homomorphism $P \to G$. Its kernel is the free group generated by the set of commutators $[c,x] = c^{-1}x^{-1}cx \in P$ with $c \in S^1 - \{1\}$ and $x \in \mathrm{SO}(3) - \{1\}$. Therefore one has a short exact sequence of topological groups $F[S^1, \mathrm{SO}(3)] \to P \xrightarrow{\pi} G$.

In a similar way we obtain another short exact sequence of topological groups $F[S^1, S^1] \to \widetilde{P} \xrightarrow{\tilde{\pi}} S^1 \times S^1 \times S^1$. Notice that $F[S^1, X] \simeq \Omega\Sigma[S^1, X] \simeq \Omega\Sigma^2 X$. Hence the quotient of the two short exact sequences yields the desired fibration (3). Although both π and $\tilde{\pi}$ have continuous sections σ and $\tilde{\sigma}$, the section σ is not a homomorphism and therefore does not immediately descend to a section of $\bar{\pi}$. However the restriction of σ to the first two factors is a group homomorphism and therefore descends to a section $\bar{\sigma}$. This implies the splitting of homotopy groups. The non-triviality of the fibration is an immediate consequence of the computation of the rational cohomology ring of $\Im\mathrm{Emb}_\omega(c,\mu)$ (see Theorem 4). □

From this theorem it follows easily that:

Theorem 3. *If* $0 < \mu - 1 \leqslant c < 1$ *then the topological space* $\mathrm{Emb}_\omega(c,\mu)$ *is weakly homotopy equivalent to the pull-back of fibration* (3) *by the fibered map* $F_\omega \to S^2 \times S^2$ *where* F_ω *is the space of symplectic frames over* $M_\mu = S^2 \times S^2$.

Then, using the minimal models of $\mathrm{Symp}(\widetilde{M}_{\mu,c})$ and $\mathrm{Symp}(M_\mu)$, we compute the minimal model of $\Im\mathrm{Emb}_\omega(c,\mu)$. This computation together with Theorem 2 gives its rational cohomology ring.

Theorem 4. *The minimal model of* $\Im\mathrm{Emb}_\omega(c,\mu)$ *is given by*

$$\Lambda\big(\Im\mathrm{Emb}_\omega(c,\mu)\big) = \Lambda(a,b,e,f,g,h) = \Lambda(S^2 \times S^2) \otimes \Lambda(g,h)$$

with generators in degrees 2, 2, 3, 3, 3, 4 *and with differential* $de = a^2$, $df = b^2$, $dg = da = db = 0$, $dh = kbg$, *where* $\Lambda(S^2 \times S^2)$ *is the minimal model for* $S^2 \times S^2$ *and* k *is a non-zero rational number.*

Then the rational cohomology ring of $\Im\mathrm{Emb}_\omega(c,\mu)$ *is equal to the algebra*

$$H^*\big(\Im\mathrm{Emb}_\omega(c,\mu);\mathbb{Q}\big) = \Lambda(a,b,c,gh,\dots,gh^n,\dots,bh,\dots,bh^n,\dots)/\langle a^2, b^2, bg\rangle$$

where $n \in \mathbb{N}$. *It is therefore not homotopy equivalent to a finite-dimensional CW-complex.*

Sketch of the proof. As first step, recall that any fibration $V \hookrightarrow P \to U$ for which the theory of minimal models applies (i.e. each space has a nilpotent homotopy system and the π_1 of the base acts trivially on the higher homotopy groups of the fiber) gives rise to a sequence

$$\big(\Lambda(U), d_U\big) \longrightarrow \big(\Lambda(U) \otimes \Lambda(V), d\big) \longrightarrow \big(\Lambda(V), d_V\big)$$

where the differential algebra in the middle is a model for the total space. We apply this theory to the fibration (1). In order to compute the differential d, we need to use different methods. For the generators a, b, g, a simple method of dimension counting gives the answer. Since the model is minimal, there is no linear term in the differential and Sullivan's duality can be expressed by $db_k = \sum_{i,j} \langle b_k, [b_i, b_j] \rangle b_i b_j$ where $\langle c_1, c_2 \rangle$ denotes the c_1-coefficient in the expression of c_2, and where the brackets denote the Whitehead product.

One can show that $[a, b] = 0$, $[a, a] = e$ and $[b, b] = f$. This gives de and df. Finally, a more sophisticated argument using the Eilenberg–Moore spectral sequence shows that dh does not vanish (see [2] for details and acknowledgments). A careful comparison of the Serre spectral sequence of fibration (3) and the minimal model computation gives the rational cohomology ring of $\Im\mathrm{Emb}_\omega(c, \mu)$. $\square$

A simple calculation using the fibration $U(2) \to \mathrm{Emb}_\omega(c, \mu) \to \Im\mathrm{Emb}_\omega(c, \mu)$, which is the restriction to B_c of fibration (1), then yields the minimal model of $\mathrm{Emb}_\omega(c, \mu)$.

Theorem 5. *A minimal model of* $\mathrm{Emb}_\omega(c, \mu)$ *is* $(\Lambda(\tilde{d}_{a,b}, \tilde{e}, \tilde{f}, \tilde{g}, v, \tilde{h}), d_0)$ *with generators of degrees* 2, 3, 3, 3, 3, 4 *and with differential given by*

$$d_0\tilde{d}_{a,b} = d_0\tilde{f} = d_0\tilde{g} = d_0v = 0, \quad d_0\tilde{e} = \tilde{d}_{a,b}^2 \quad \text{and} \quad d_0\tilde{h} = -k\tilde{d}_{a,b}\tilde{g}$$

where k is a non-zero rational number.

See [2] for the computation of the cohomology rings of $\Im\mathrm{Emb}_\omega(c, \mu)$ and $\mathrm{Emb}_\omega(c, \mu)$ with any field coefficients.

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