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Numerical Analysis

The symmetric discontinuous Galerkin method does not need stabilization in 1D for polynomial orders $p \geqslant 2$

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Abstract

In this Note we prove that in one space dimension, the symmetric discontinuous Galerkin method for second order elliptic problems is stable for polynomial orders $p \geqslant 2$ without using any stabilization parameter. The method yields optimal convergence rates in both the energy norm (L^2-norm of broken gradient plus jump terms) and the L^2-norm and can be written in conservative form with fluxes independent of any stabilization parameter. ***To cite this article: E. Burman et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

La méthode de Galerkine discontinue symétrique est stable en une dimension d'espace pour tout ordre polynômial $p \geqslant 2$. Dans cette Note, nous montrons qu'en une dimension d'espace, la méthode de Galerkine discontinue symétrique pour les problèmes elliptiques d'ordre deux est stable pour tout ordre polynômial $p \geqslant 2$ sans devoir introduire de paramètre de stabilisation. La méthode fournit des ordres de convergence optimaux dans la norme d'énergie (norme L^2 du gradient brisé plus des termes de saut) et dans la norme L^2 et peut être écrite sous forme conservative avec des flux indépendants de tout paramètre de stabilisation. ***Pour citer cet article : E. Burman et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction

The Discontinuous Galerkin (DG) method is a classical technique to approximate elliptic and hyperbolic PDE's. A unified theory has been developed recently in the framework of Friedrichs' systems [4]. For elliptic PDE's, two of the most popular methods are the Symmetric Interior Penalty (SIP) method introduced by Baker [2] and Arnold [1]

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and the Nonsymmetric DG method introduced by Oden, Babuška and Baumann [7]. One attractive feature of the latter method is that, because of the absence of penalty terms, it can be written in conservative form with fluxes that are independent of numerical parameters. Moreover, the Nonsymmetric DG method has been proven to yield optimal convergence estimates in the energy semi-norm (L^2-norm of broken gradient) on triangles, parallelograms and tetrahedra for polynomial orders $p \geqslant 2$ [8]. The inf-sup stability and optimal convergence in the energy norm (L^2-norm of broken gradient plus jump terms) have been established in one dimension [5] and on triangles in two dimensions [6], still for polynomial orders $p \geqslant 2$. For $p = 1$, penalty terms must be introduced to grant stability and optimal convergence rates, but the conservative fluxes then depend on the penalty parameter.

Working with the SIP method instead of the Nonsymmetric DG method presents the twofold advantage of dealing with symmetric linear systems and of ensuring optimal convergence rates also in the L^2-norm. The difficulty with the SIP method is that stability usually relies on the use of penalty parameters that will subsequently enter the expression of the conservative fluxes. An exception was provided in the case of polynomial order $p = 1$ in [3] where stable SIP methods were proposed in two and three space dimensions without stabilization on interior faces.

The purpose of this Note is to fill the gap between Symmetric and Nonsymmetric DG methods in one space dimension. We indeed prove that the symmetric DG method without any penalty leads to optimal convergence rates in the energy norm and in the L^2-norm in one space dimension for polynomial orders $p \geqslant 2$. One relevant difference with the Nonsymmetric DG method still remains, namely that although the Symmetric DG method is proven here to be inf-sup stable, the stiffness matrix can have negative eigenvalues, a fact that can be important in time-dependent problems.

2. Model problem and method formulation

Let $\Omega = (a, b) \subset \mathbb{R}$, $f \in L^2(\Omega)$ and $g_a, g_b \in \mathbb{R}$. Consider the following boundary value problem:

$$-u'' = f \quad \text{in } \Omega, \qquad u(a) = g_a, \quad u(b) = g_b. \tag{1}$$

This problem is well-posed in $H^1(\Omega)$ and since $f \in L^2(\Omega)$, its unique solution is in $H^2(\Omega)$. Let $\mathcal{K}_h$ be a partition of the domain Ω formed by M elements $K_i = (x_{i-1}, x_i)$. For simplicity, $\mathcal{K}_h$ is assumed to be uniform, i.e., $x_i = a + ih$, $i \in \{0, \ldots, M\}$ where $h = \frac{b-a}{M}$ denotes the mesh size. Let an integer $p \geqslant 0$ and consider the usual discontinuous finite element space

$$V_h^p = \{v \in L^2(\Omega);\ \forall i \in \{1, \ldots, M\},\ v|_{K_i} \in \mathbb{P}_p(K_i)\}, \tag{2}$$

where $\mathbb{P}_p(K_i)$ denotes the p-th order polynomial space on K_i. Let $\mathcal{N}_h$ denote the set of all nodes of $\mathcal{K}_h$ and let $\mathcal{N}_h^{\mathrm{i}}$ denote the set of all interior nodes. For any function $v \in H^1(\mathcal{K}_h)$, where for any $s \geqslant 1$, $H^s(\mathcal{K}_h)$ denotes the usual broken Sobolev space of order s, define its jump and average at interior nodes as follows:

$$[\![v]\!]_i = v|_{K_i}(x_i) - v|_{K_{i+1}}(x_i), \qquad \{v\}_i = \frac{1}{2}\big(v|_{K_{i+1}}(x_i) + v|_{K_i}(x_i)\big).$$

On boundary nodes, the following notation is used: $[\![v]\!]_0 = -v(a)$, $[\![v]\!]_M = v(b)$, $\{v\}_0 = v(a)$ and $\{v\}_M = v(b)$. For any region $R \subset \Omega$ composed of one or more mesh cells, $(\cdot,\cdot)_R$ denotes the usual $L^2(R)$-scalar product and $\|\cdot\|_R$ the associated norm. In the sequel, the inequality $A \lesssim B$ means that there is a positive c, independent of h, such that $A \leqslant cB$. For simplicity, the dependency of the constants on p is not addressed herein.

The symmetric DG method consists of finding $u_h \in V_h^p$ such that

$$a_h(u_h, v_h) = l(v_h), \quad \forall v_h \in V_h^p \tag{3}$$

where

$$a_h(u_h, v_h) = \sum_{K_i \in \mathcal{K}_h} (u_h', v_h')_{K_i} - \sum_{x_i \in \mathcal{N}_h} \big([\![u_h]\!]_i \{v_h'\}_i + \{u_h'\}_i [\![v_h]\!]_i\big), \tag{4}$$

$$l(v_h) = \sum_{K_i \in \mathcal{K}_h} (f, v_h)_{K_i} + g_a v_h'(a) - g_b v_h'(b). \tag{5}$$

Observe that the jumps of the discrete solution at interior nodes are not penalized and that the boundary conditions are not enforced by penalty but just through the consistency terms, i.e., the contribution of boundary nodes in the last term of Eq. (4). Furthermore, the discrete problem (3) is consistent. Indeed, integration by parts yields for any $v, w \in H^2(\mathcal{K}_h)$,

$$a_h(v, w) = -\sum_{K_i \in \mathcal{K}_h} (v'', w)_{K_i} - \sum_{x_i \in \mathcal{N}_h} [\![v]\!]_i \{w'\}_i + \sum_{x_i \in \mathcal{N}_h^{\mathrm{i}}} [\![v']\!]_i \{w\}_i. \tag{6}$$

Applying this with $v := u$, the solution to (1), and $w := v_h$ arbitrary in V_h^p yields $a_h(u, v_h) = l(v_h)$.

3. Convergence analysis

Define the following energy norm in $H^1(\mathcal{K}_h)$:

$$|||v|||^2 = \|v'\|_{\mathcal{K}_h}^2 + \sum_{x_i \in \mathcal{N}_h} \frac{1}{h} [\![v]\!]_i^2 \quad \text{where } \|v'\|_{\mathcal{K}_h}^2 = \sum_{K_i \in \mathcal{K}_h} \|v'\|_{K_i}^2.$$

The main technical result of this section is the following:

Lemma 3.1. *Assume $p \geqslant 2$. Then,*

$$\forall v_h \in V_h^p, \quad |||v_h||| \lesssim \sup_{0 \neq w_h \in V_h^p} \frac{a_h(v_h, w_h)}{|||w_h|||}. \tag{7}$$

Proof. Let $v_h \in V_h^p$.

(i) Let us prove that there is (a unique) $y_h \in V_h^p$ such that

$$\begin{cases} (y_h, z_h)_\Omega = 0, & \forall z_h \in V_h^{p-2}, \\ \{y_h'\}_i = \frac{1}{h}[\![v_h]\!]_i, & \forall i \in \{0, \dots, M\}, \\ \{y_h\}_i = 0, & \forall i \in \{1, \dots, M-1\}. \end{cases} \tag{8}$$

To this purpose, let us first establish the a priori estimate

$$|||y_h||| \lesssim |||v_h|||. \tag{9}$$

Since $y_h \perp V_h^{p-2}$ and $p \geqslant 2$, y_h has zero mean over each mesh cell. As a result, y_h satisfies for all $i \in \{1, \dots, M\}$, the strong Poincaré inequality

$$\|y_h\|_{K_i} \lesssim h \|y_h'\|_{K_i}.$$

Hence, using a trace inequality yields

$$\sum_{x_i \in \mathcal{N}_h} \frac{1}{h} [\![y_h]\!]_i^2 \lesssim \frac{1}{h^2} \sum_{K_i \in \mathcal{K}_h} \|y_h\|_{K_i}^2 \lesssim \|y_h'\|_{\mathcal{K}_h}^2.$$

Moreover, integrating by parts and using the properties of y_h, it is inferred that

$$\begin{aligned} \|y_h'\|_{\mathcal{K}_h}^2 &= -\sum_{K_i \in \mathcal{K}_h} (y_h'', y_h)_{K_i} + \sum_{x_i \in \mathcal{N}_h^{\mathrm{i}}} [\![y_h']\!]_i \{y_h\}_i + \sum_{x_i \in \mathcal{N}_h} [\![y_h]\!]_i \{y_h'\}_i \\ &= \sum_{x_i \in \mathcal{N}_h} [\![y_h]\!]_i \frac{1}{h} [\![v_h]\!]_i \lesssim |||v_h||| \left(\sum_{x_i \in \mathcal{N}_h} \frac{1}{h} [\![y_h]\!]_i^2 \right)^{1/2} \lesssim |||v_h||| \|y_h'\|_{\mathcal{K}_h}, \end{aligned}$$

whence the a priori estimate (9) readily follows. To conclude this first step of the proof, it now suffices to observe that (8) is nothing more than a square linear system of size $(p+1)M$. Hence, the existence of y_h is equivalent to the fact that the matrix associated with (8) has zero kernel, which, in turn, is a straightforward consequence of the a priori estimate (9).

(ii) Owing to (6) and (8), $a_h(v_h, -y_h) = \sum_{x_i \in \mathcal{N}_h} \frac{1}{h} [\![v_h]\!]_i^2$. Furthermore, using a trace inequality leads to

$$a_h(v_h, v_h) = \|v_h'\|_{\mathcal{K}_h}^2 - 2 \sum_{x_i \in \mathcal{N}_h} [\![v_h]\!]_i \{v_h'\}_i \geqslant \frac{1}{2} \|v_h'\|_{\mathcal{K}_h}^2 - c \sum_{x_i \in \mathcal{N}_h} \frac{1}{h} [\![v_h]\!]_i^2,$$

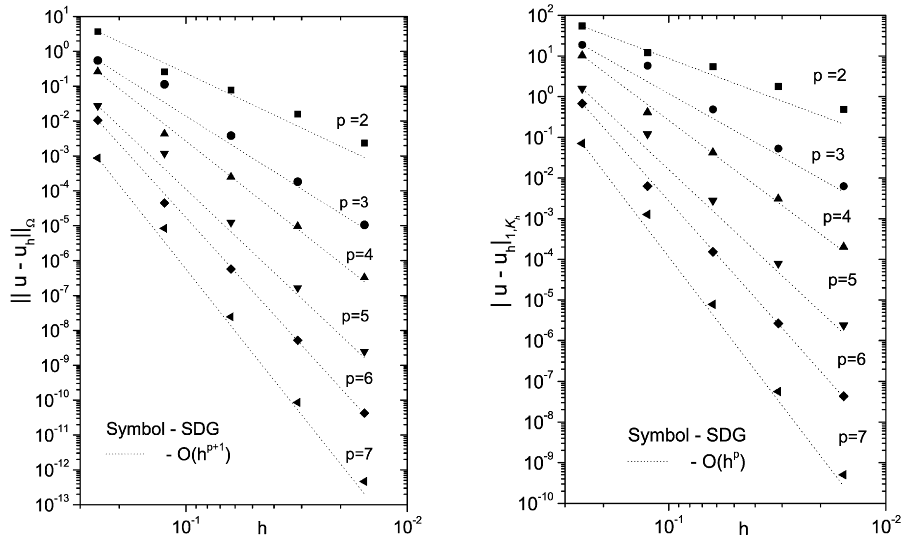

Fig. 1. $L^2(\Omega)$ (left) and $H^1(\mathcal{K}_h)$ (right) semi-norm of the error as function of the mesh diameter h.

with c independent of h. Hence, there is λ large enough such that $|||v_h|||^2 \lesssim a_h(v_h, v_h - \lambda y_h)$, whence (7) is readily inferred owing to (9). □

Observe that the proof of Lemma 3.1 breaks down for $p = 1$ because it cannot be inferred that y_h has zero mean elementwise and thus the strong Poincaré inequality cannot be used. A direct analysis shows that the matrix associated with the bilinear form a_h on V_h^1 is singular with one-dimensional kernel spanned by the so-called checkerboard mode (the function in V_h^0 equal to ± 1 on alternating mesh cells). This matrix becomes nonsingular if the bilinear form a_h is supplemented by penalizing a jump at an interior node or one boundary value. In the multidimensional case with $p = 1$, the checkerboard mode can be controlled by mesh geometry [3,6].

Theorem 3.2. *Let $u \in H^r(\mathcal{K}_h) \bigcap H^2(\Omega)$, $r \geqslant 2$, solve* (1) *and let $u_h \in V_h^p$, $p \geqslant 2$, solve* (3). *Then, for all $2 \leqslant s \leqslant \min(p+1, r)$,*

$$\|u - u_h\|_\Omega + h|||u - u_h||| \leqslant ch^s \|u\|_{H^s(\mathcal{K}_h)}.$$

Proof. Use (7) and standard finite element techniques. □

To illustrate, consider $\Omega = (0, 1)$ with homogeneous boundary conditions and with f such that the solution is $u(x) = \sin(12\pi x)e^{1.75x}$. Fig. 1 presents the convergence rates in the $H^1(\mathcal{K}_h)$- and $L^2(\Omega)$-norms for a sequence of nested uniform meshes and for approximation orders $p \in \{2, \ldots, 7\}$.

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