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Partial Differential Equations

On moderately close inclusions for the Laplace equation

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Abstract

The presence of small inclusions modifies the solution of the Laplace equation posed in a reference domain Ω_0. This question has been widely studied for a single inclusion or well-separated inclusions. We investigate in this Note the case where the distance between the holes tends to zero but remains large with respect to their characteristic size. We first consider two perfectly insulated inclusions. In this configuration we give a complete multiscale asymptotic expansion of the solution to the Laplace equation. We also address the situation of a single inclusion close to a singular perturbation of the boundary $\partial\Omega_0$. ***To cite this article: V. Bonnaillie-Noël et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Interactions entre inclusions relativement proches pour l'équation de Laplace. La présence de petites inclusions dans un domaine de référence Ω_0 modifie la solution de l'équation de Laplace dans ce domaine. Les cas d'une inclusion isolée ou de plusieurs bien séparées ont été largement étudiés. Dans cette Note, nous considérons le cas où la distance entre deux inclusions tend vers zéro mais reste grande par rapport à leur taille caractéristique. Nous donnons un développement asymptotique multi-échelle complet de la solution de l'équation de Laplace dans la situation de deux inclusions parfaitement isolantes. Nous présentons également le cas d'une seule inclusion proche du bord $\partial\Omega_0$ qui est lui même perturbé. ***Pour citer cet article : V. Bonnaillie-Noël et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

Soit Ω_0 un domaine borné de $\mathbb{R}^2$. Pour $\varepsilon > 0$ suffisamment petit, on considère le domaine perturbé Ω_ε obtenu en enlevant à Ω_0 deux inclusions de taille ε et séparées d'une distance $2\eta_\varepsilon$. Précisément le domaine Ω_ε est défini par la relation (1), où les motifs $\omega^\pm$, contenant l'origine, sont dilatés à l'échelle ε, et translatés en $x_\varepsilon^\pm = 0 \pm \eta_\varepsilon \mathbf{d}$, voir Fig. 1. Notre objectif est de donner un développement asymptotique de la solution du problème aux limites (2) posé dans Ω_ε.

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Le cas d'une seule inclusion a été largement étudié dans [8,6,7,9,3,4,1,5]. Ces travaux s'appuient sur la notion essentielle de *profil*, solution normalisée de l'équation de Laplace dans le domaine extérieur obtenu par *blow-up* de la perturbation (voir (4)). Exprimé en variable rapide x/ε, le profil V_0 décrit le comportement local de la solution au voisinage de l'inclusion. La convergence du développement est établie grâce à la décroissance de V_0 à l'infini. Ainsi, dans le cas de conditions aux limites de type Neumann sur le bord de l'inclusion, le début du développement est donné par $u_\varepsilon(x) = u_0(x) + \varepsilon V_0(x/\varepsilon) + r_\varepsilon^1(x)$, où u_0 est la solution de l'équation de Laplace dans Ω_0 et V_0 résout (4). Le reste satisfait $\|r_\varepsilon^1\|_{\mathrm{H}^1(\Omega_\varepsilon)} = \mathcal{O}(\varepsilon^2)$.

Cette étude s'étend directement aux cas $\eta_\varepsilon = \varepsilon$ (un unique motif composé des deux inclusions) et $\eta_\varepsilon = \mathcal{O}(1)$ (deux inclusions indépendantes, voir aussi [8, §5.3]). L'objet de cette note est de décrire les régimes intermédiaires obtenus lorsque $\eta_\varepsilon = \varepsilon^\alpha$ avec $\alpha \in (0, 1)$.

Dans le cas de deux inclusions (avec conditions de Neumann) centrées en x_ε^- et x_ε^+ distants de $2\varepsilon^\alpha$, le début du développement est donné par (8). Notons que les profils $V_0^\pm$ qui interviennent ici ne sont pas adaptés à la géométrie locale des cas limites $\alpha \to 0$ et $\alpha \to 1$. L'ordre du développement est alors détérioré. On peut interpréter (8) de la façon suivante : la contribution principale des deux inclusions est due à la superposition de leur effet individuel. Le reste r_ε^1 contient l'information d'ordre supérieur :

- pour $\alpha < 2/3$, les inclusions sont relativement éloignées l'une de l'autre. Le terme dominant du reste est d'ordre $\mathcal{O}(\varepsilon^{1+\alpha})$ et provient du développement de Taylor de u_0 en 0 ;
- pour $2/3 < \alpha < 1$, les inclusions sont plus proches. Le reste r_ε^1 est d'ordre $\mathcal{O}(\varepsilon^{3-2\alpha})$, il rend compte principalement de l'*interaction* entre les profils V_0^- et V_0^+.

Les techniques développées s'appliquent encore si l'une des inclusions est située au bord du domaine Ω_0. Toutefois, pour des raisons techniques, on doit introduire des fonctions de troncature pour assurer la superposition des différents termes.

Les développements multi-échelle complets dans chaque cas sont détaillés dans les Théorèmes 2.1 et 2.2.

1. Introduction

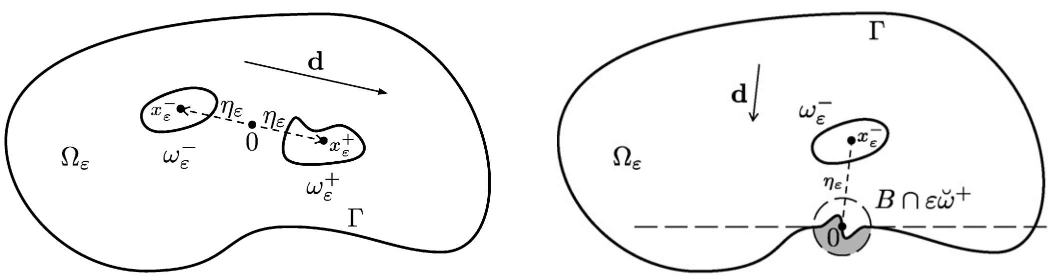
Let Ω_0, ω^-, and ω^+ be three bounded domains of $\mathbb{R}^2$, each containing the origin 0. For $\varepsilon > 0$, small enough, we define the perturbed domain Ω_ε as

$$\Omega_\varepsilon = \Omega_0 \setminus \overline{(\omega_\varepsilon^- \cup \omega_\varepsilon^+)}, \quad \text{with } \omega_\varepsilon^\pm = x_\varepsilon^\pm + \varepsilon\omega^\pm, \tag{1}$$

where $x_\varepsilon^\pm = \pm\eta_\varepsilon \mathbf{d}$ with a given unitary vector $\mathbf{d}$, and a real number η_ε. Shortly, Ω_ε consists of Ω_0 from which two ε-inclusions at distance $2\eta_\varepsilon$ have been removed, cf. Fig. 1.

We aim at building an asymptotic expansion of the solution u_ε of the Laplace problem in Ω_ε

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \Gamma = \partial\Omega_0, \\ \partial_{\mathbf{n}} u_\varepsilon = 0 & \text{on } \partial\omega_\varepsilon^\pm, \end{cases} \tag{2}$$



(a) Two interior inclusions of size ε, at distance $2\eta_\varepsilon$. (b) Boundary perturbation.

Fig. 1. Geometrical settings for perturbed domains.

Fig. 1. Géométrie des domaines perturbés.

for some L^2 datum f whose support does not contain the origin 0. We restrict ourselves to homogeneous Neumann boundary conditions on $\partial\omega_\varepsilon^\pm$, although generalizations to other conditions are possible. Besides, one of the inclusions may be localized at the boundary Γ of Ω_0 (or even simply be removed, the remaining inclusion moving towards the external boundary). The origin no longer belongs to the domains themselves, but only to their closure.

The case of a single inclusion ω, centered at the origin 0 being either in Ω_0 or in Γ, has been widely studied, see [8,6,7,9,3,4,1,5]. The techniques rely on the notion of *profile*, a normalized solution of the Laplace equation in the exterior domain obtained by *blow-up* of the perturbation, see (4). It is used in a fast variable to describe the local behavior of the solution in the perturbed domain. Convergence of the asymptotic expansion is obtained thanks to the decay of the *profile* at infinity. For example, if we impose Neumann boundary conditions on the inclusion, the expansion takes the form

$$u_\varepsilon(x) = u_0(x) + \varepsilon V_0\left(\frac{x}{\varepsilon}\right) + r_\varepsilon^1(x), \qquad \text{with } \|r_\varepsilon^1\|_{H^1(\Omega_\varepsilon)} = \mathcal{O}(\varepsilon^2), \tag{3}$$

where

- u_0 is the solution of the Laplace–Dirichlet problem in Ω_0: $u_0 \in H_0^1(\Omega_\varepsilon)$, $-\Delta u_0 = f$,
- V_0 is a *profile* satisfying

$$\begin{cases} -\Delta V_0 = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\omega}, \\ \partial_{\mathbf{n}} V_0 = -\nabla u_0(0)\cdot \mathbf{n} & \text{on } \partial\omega, \\ V_0 \to 0 & \text{at infinity.} \end{cases} \tag{4}$$

The previous results can easily be extended to the case of two (or finitely many) inclusions within two situations:

1. *Inclusions at distance $\mathcal{O}(1)$*. It corresponds to $\eta_\varepsilon = \eta$ independent of ε. In this case, the centers $x^\pm$ are independent of ε. The decaying profiles $V_0^\pm$ are harmonic in $\mathbb{R}^2 \setminus \overline{\omega}^\pm$ and satisfy the boundary conditions

$$\partial_{\mathbf{n}} V_0^\pm = -\nabla u_0(x^\pm)\cdot \mathbf{n} \quad \text{on } \partial\omega^\pm.$$

At the first order, the holes do not interact with each other, their contributions are merely superposed

$$u_\varepsilon(x) = u_0(x) + \varepsilon\left[V_0^+\left(\frac{x-x^+}{\varepsilon}\right) + V_0^-\left(\frac{x-x^-}{\varepsilon}\right)\right] + r_\varepsilon^1(x), \quad \text{with } \|r_\varepsilon^1\|_{H^1(\Omega_\varepsilon)} = \mathcal{O}(\varepsilon^2). \tag{5}$$

2. *Inclusions at distance $\mathcal{O}(\varepsilon)$*. It corresponds to $\eta_\varepsilon = c\,\varepsilon$ with a constant $c \in \mathbb{R}$. Here the two inclusions constitute a unique pattern at the scale ε. This case is actually handled as a single inclusion $\omega = \omega^+ \cup \omega^-$, selfsimilar with respect to the origin 0. The expansion reads

$$u_\varepsilon(x) = u_0(x) + \varepsilon W_0\left(\frac{x}{\varepsilon}\right) + r_\varepsilon^1(x), \qquad \text{with } \|r_\varepsilon^1\|_{H^1(\Omega_\varepsilon)} = \mathcal{O}(\varepsilon^2), \tag{6}$$

where the profile W_0 is associated with the whole pattern ω.

These two situations show radically different behaviors: no interaction and full interaction. We focus in this work on the intermediate cases, where the inclusions are *moderately close*, i.e.

$$\eta_\varepsilon \to 0 \quad \text{and} \quad \eta_\varepsilon/\varepsilon \to +\infty \quad (\text{as } \varepsilon \to 0). \tag{7}$$

One can expect to have a weak interaction between the two inclusions. To quantify this effect, we specify the range η_ε as $\eta_\varepsilon = \varepsilon^\alpha$ with $\alpha \in (0,1)$. The limit case $\alpha = 0$ corresponds to inclusions at distance $\mathcal{O}(1)$ while the other limit $\alpha = 1$ corresponds to inclusions at distance $\mathcal{O}(\varepsilon)$.

2. Multiscale asymptotic expansions

We now consider the situation of Fig. 1, where the distance between the two inclusions equals ε^α with $\alpha \in (0,1)$, and we focus on the following two-dimensional problems which cover the main difficulties and techniques: $u_\varepsilon \in H^1(\Omega_\varepsilon)$ satisfies the Laplace equation $-\Delta u_\varepsilon = f$ with various boundary conditions, see Fig. 1:

(a) *two Neumann inclusions*: $u_\varepsilon = 0$ on Γ and $\partial_{\mathbf{n}} u_\varepsilon = 0$ on $\partial\omega_\varepsilon^- \cup \partial\omega_\varepsilon^+$,
(b) *a Neumann inclusion and a Dirichlet boundary perturbation*:[1] $\partial_{\mathbf{n}} u_\varepsilon = 0$ on $\partial\omega_\varepsilon^-$, $u_\varepsilon = 0$ elsewhere.

Let us mention that a three-scale problem has also been investigated with multiscale techniques in [8, §5.4], which consists in a bump at scale $\varepsilon^{1+\kappa}$ on a ε-boundary singular perturbation of a smooth domain.

We start with giving a brief description of the first terms in the expansions. Theorems 2.1 and 2.2 state the complete asymptotics with optimal remainder estimates.

Case (a). For two Neumann inclusions, centered respectively in x_ε^- and x_ε^+ (separated by a distance $2\varepsilon^\alpha$), the first correctors involve the profiles $V_0^\pm$ as introduced in (4)

$$u_\varepsilon(x) = u_0(x) + \varepsilon\left[V_0^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + V_0^+\left(\frac{x - x_\varepsilon^+}{\varepsilon}\right)\right] + r_\varepsilon^1(x), \quad \text{with } \|r_\varepsilon^1\|_{\mathrm{H}^1(\Omega_\varepsilon)} = \mathcal{O}(\varepsilon^{\min(1+\alpha, 3-2\alpha)}). \tag{8}$$

The profiles satisfy $\| V_0^\pm(\frac{\cdot - x_\varepsilon^-}{\varepsilon})\|_{\mathrm{H}^1(\Omega_\varepsilon)} = \mathcal{O}(1)$, they only depend on the shape of $\omega^\pm$ and on the gradient of the limit term at the origin $\nabla u_0(0)$. We emphasize that the origins $x_\varepsilon^\pm$ of the profiles do vary with ε, unlike $x^\pm$ in equation (5) and 0 in (6). Moreover the remainder as $\alpha \to 0$ or $\alpha \to 1$ is of order ε because of the inadequation of the profiles with the geometry.

We may understand expansion (8) in the following way: the main contribution of the two inclusions is merely the superposition of their individual effect. The remainder r_ε^1 contains information about higher order influence. It is interesting to describe further the structure of this remainder:

- for $\alpha < 2/3$, the inclusions are relatively far away from each other. The leading term in r_ε^1 is $\mathcal{O}(\varepsilon^{1+\alpha})$ and arises from the Taylor expansion of u_0 at the origin 0;
- for $2/3 < \alpha < 1$, the inclusions are closer. The remainder r_ε^1 is $\mathcal{O}(\varepsilon^{3-2\alpha})$ and mainly consists in the *interaction* between the profiles V_0^- and V_0^+.

Theorem 2.1. *The solution u_ε of problem* (2) *admits the expansion at order n*

$$\begin{aligned} u_\varepsilon(x) = {} & u_0(x) + \varepsilon\left[V_0^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + V_0^+\left(\frac{x - x_\varepsilon^+}{\varepsilon}\right)\right] \\ & + \varepsilon\left[\varepsilon^\alpha\left[V_\alpha^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + V_\alpha^+\left(\frac{x - x_\varepsilon^+}{\varepsilon}\right)\right] + \varepsilon\left[V_1^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + V_1^+\left(\frac{x - x_\varepsilon^+}{\varepsilon}\right)\right]\right] \\ & + \sum_{(p,q)\in K_n} \varepsilon^{p+\alpha q} v_{p+\alpha q}(x) + \varepsilon \sum_{(p,q)\in K_n} \varepsilon^{p+\alpha q}\left[V_{p+\alpha q}^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + V_{p+\alpha q}^+\left(\frac{x - x_\varepsilon^+}{\varepsilon}\right)\right] + r_\varepsilon^n(x), \end{aligned}$$

with $K_n = \{(p,q) \in \mathbb{Z}^2 \mid p \geqslant 0,\ q \geqslant -\frac{3}{2}p + 2,\ q \geqslant -p$ and $p + \alpha q \leqslant n\}$ and $\|r_\varepsilon^n\|_{\mathrm{H}^1(\Omega_\varepsilon)} = o(\varepsilon^n)$.

The terms arising in the previous expansion are generated in three different ways:

- *defect of Neumann boundary condition for u_0*: the terms in the Taylor expansion of the limit term u_0 at the origin 0 are successively corrected by profiles in the fast variable. Since the boundary of the inclusions are at distance $\mathcal{O}(\varepsilon^\alpha + \varepsilon)$, powers of ε^α and ε are both involved;
- *interaction between the two inclusions*: the profile V_0^+ does not satisfy Neumann conditions on $\partial\omega_\varepsilon^-$ (and conversely), we need profiles which correct the normal derivative;
- *interaction between the inclusions and the external boundary*: the Dirichlet condition is not strictly fulfilled by the profiles, which generates slow variable correctors.

[1] In this case, the definition of the perturbed domain Ω_ε is slightly different, see [5].

Case (b). This situation requires a slightly different definition of the geometry: the origin 0 is assumed to be on the boundary Γ of Ω_0, and Γ to coincide with a straight line in a neighborhood of 0. The perturbed domain Ω_ε is defined as

$$\Omega_\varepsilon = \left[\Omega_0 \backslash (\overline{\omega_\varepsilon^-} \cup B)\right] \cup (B \cap \varepsilon\breve{\omega}^+), \tag{9}$$

where B is a small (but fixed with respect to ε) ball centered in 0 and $\breve{\omega}^+$ is a perturbed upper half plane.

The boundary of Ω_0 is perturbed in such a way that $\Omega_\varepsilon \not\subset \Omega_0$, we hence need cut-off functions (see [5,11]). Precisely, the asymptotic expansion takes the form

$$u_\varepsilon(x) = \zeta\left(\left|\frac{x}{\varepsilon}\right|\right)u_0(x) + \varepsilon\left[V_0^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + \chi\left(|x|\right)V_0^+\left(\frac{x}{\varepsilon}\right)\right] + r_\varepsilon^1(x) \quad \text{with } \|r_\varepsilon^1\|_{\mathrm{H}^1(\Omega_\varepsilon)} = o(\varepsilon), \tag{10}$$

where $\zeta(r)$ vanishes for $r < r_\bullet$ and $\zeta(r) = 1$ for $r > r^\bullet$, and $\chi(r) = 1$ for $r < r_*$ and $\chi(r) = 0$ for $r > r^*$. The remarks about the interaction between the two perturbations in case (a) still hold.

Theorem 2.2. *Let K_n be defined as in Theorem* 2.1. *The solution u_ε of*

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \setminus \overline{\omega_\varepsilon^-}, \\ \partial_{\mathbf{n}} u_\varepsilon = 0 & \text{on } \partial\omega_\varepsilon^-, \end{cases} \tag{11}$$

admits the expansion at order n

$$\begin{aligned} u_\varepsilon(x) = {} & \zeta\left(\left|\frac{x}{\varepsilon}\right|\right)u_0(x) + \varepsilon\left[V_0^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + \chi\left(|x|\right)V_0^+\left(\frac{x}{\varepsilon}\right)\right] \\ & + \varepsilon\left[\varepsilon^\alpha\left[V_\alpha^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + \chi\left(|x|\right)V_\alpha^+\left(\frac{x}{\varepsilon}\right)\right] + \varepsilon\left[V_1^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + \chi\left(|x|\right)V_1^+\left(\frac{x}{\varepsilon}\right)\right]\right] \\ & + \zeta\left(\left|\frac{x}{\varepsilon}\right|\right) \sum_{(p,q)\in K_n} \varepsilon^{p+\alpha q} v_{p+\alpha q}(x) + \varepsilon \sum_{(p,q)\in K_n} \varepsilon^{p+\alpha q}\left[V_{p+\alpha q}^-\left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + \chi\left(|x|\right)V_{p+\alpha q}^+\left(\frac{x}{\varepsilon}\right)\right] \\ & + r_\varepsilon^n(x), \end{aligned}$$

with

$$\|r_\varepsilon^n\|_{\mathrm{H}^1(\Omega_\varepsilon)} = o(\varepsilon^n).$$

More details and proofs will be available in [2].

3. Concluding remarks

The previous results extend to other situations:

- *three-dimensional domains*: the construction of the asymptotics goes through, decaying profiles are also involved and the structure of the expansion remains the same;
- *perturbation of a curved boundary*: in Theorem 2.2, we assumed the boundary Γ to be flat near the perturbation point 0. This assumption cannot be easily removed since the profile V_0^+ is defined on a geometry which does not fit the local curvature. To get round this difficulty, it is possible to consider perturbation which are self-similar after straightening, see [10]. For a perturbation self-similar in the physical coordinates in a locally convex domain, an expansion has been obtained up to order 2 in [5];
- the case of Dirichlet conditions on the inclusions in dimension 2 requires a more evolved analysis because of the logarithmic potential. Non-decaying profiles appear, and the expansion also contains powers of $\log\varepsilon$.

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