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Partial Differential Equations

# $\Gamma$-convergence and Sobolev norms

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**Abstract**

We study a $\Gamma$-convergence problem related to a new characterization of Sobolev spaces $W^{1,p}(\mathbb{R}^N)$ $(p > 1)$ established in H.-M. Nguyen [H.-M. Nguyen, Some new characterizations of Sobolev spaces, J. Funct. Anal. 237 (2006) 689–720] and J. Bourgain and H.-M. Nguyen [J. Bourgain, H.-M. Nguyen, A new characterization of Sobolev spaces, C. R. Acad. Sci. Paris, Ser. I 343 (2006) 75–80]. We can also handle the case $p = 1$ which was out of reach previously. ***To cite this article: H.-M. Nguyen, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***



**Résumé**

***Γ*-convergence et normes de Sobolev.** On étudie un problème de $\Gamma$-convergence qui apparaît naturellement en liaison avec les travaux de H.-M. Nguyen [H.-M. Nguyen, Some new characterizations of Sobolev spaces, J. Funct. Anal. 237 (2006) 689–720], et J. Bourgain et H.-M. Nguyen [J. Bourgain, H.-M. Nguyen, A new characterization of Sobolev spaces, C. R. Acad. Sci. Paris, Ser. I 343 (2006), 75–80] concernant des nouvelles caractérisations des espaces de Sobolev $W^{1,p}(\mathbb{R}^N)$ $(p > 1)$. On peut aussi traiter le cas $p = 1$ qui était inaccessible précédemment. ***Pour citer cet article : H.-M. Nguyen, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***



## Version française abrégée

Soient $p \geqslant 1$ et $\delta > 0$. Posons

$$I_\delta(g) = \iint\limits_{\substack{\mathbb{R}^N \times \mathbb{R}^N \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy, \quad \forall g \in L^p(\mathbb{R}^N).$$

Ci-après $|\ |$ désigne la norme euclidienne de $\mathbb{R}^N$. Récemment la caractérisation suivante des espaces de Sobolev a été établie dans [10, Théorème 2] et [3, Théorème 1] :

**Théorème 1.** *Soient $N \geqslant 1$, $1 < p < +\infty$, et $g \in L^p(\mathbb{R}^N)$. Alors $g \in W^{1,p}(\mathbb{R}^N)$ si et seulement si*

$$\liminf_{\delta \to 0_+} I_\delta(g) < +\infty.$$

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*De plus*

$$\lim_{\delta\to 0_+} I_\delta(g) = \frac{1}{p} K_{N,p} \int_{\mathbb{R}^N} |Dg|^p \, dx, \quad \forall g \in W^{1,p}(\mathbb{R}^N),$$

*où* $K_{N,p}$ *est définie par*

$$K_{N,p} = \int_{\mathbb{S}^{N-1}} |e \cdot \sigma|^p \, d\sigma,$$

*pour tout* $e \in \mathbb{S}^{N-1}$.

Rappelons aussi que lorsque $p = 1$, on a

(a) Si $g \in L^1(\mathbb{R}^N)$ et $\liminf_{\delta\to 0_+} I_\delta(g) < +\infty$, alors $g \in BV(\mathbb{R}^N)$ (voir [3,12]).
(b) $\exists g \in W^{1,1}(\mathbb{R})$ telle que $\lim_{\delta\to 0_+} I_\delta(g) = +\infty$ (exemple communiqué par A. Ponce ; voir [10]).

Le résultat principal de cette Note est

**Théorème 2.** *Soient* $p \geqslant 1$ *et* $N \geqslant 1$. *Alors* $(I_\delta)$ $\Gamma$*-converge dans* $L^p(\mathbb{R}^N)$ *quand* $\delta$ *tend vers* 0 *vers la fonctionelle* $I$ *définie sur* $L^p(\mathbb{R}^N)$ *par*

$$I(g) = \begin{cases} C_{N,p} \int_{\mathbb{R}^N} |Dg|^p \, dx & \text{si } p > 1 \text{ et } g \in W^{1,p}(\mathbb{R}^N) \text{ (resp. } p = 1 \text{ et } g \in BV(\mathbb{R}^N)), \\ +\infty & \text{sinon.} \end{cases}$$

*Ici, la constante* $C_{N,p}$ *est définie par* (2) *et vérifie* $0 < C_{N,p} < \frac{1}{p} K_{N,p}$.

## 1. Introduction and the main result

For $p \geqslant 1$ and $\delta > 0$, define

$$I_\delta(g) = \iint_{\substack{\mathbb{R}^N \times \mathbb{R}^N \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy, \quad \forall g \in L^p(\mathbb{R}^N).$$

Hereafter $|\ |$ denotes the Euclidean norm of $\mathbb{R}^N$. Recently, the following new characterization of Sobolev spaces was established in [10, Theorem 2] and [3, Theorem 1]:

**Theorem 1.** *Let* $N \geqslant 1$, $1 < p < +\infty$, *and* $g \in L^p(\mathbb{R}^N)$. *Then* $g \in W^{1,p}(\mathbb{R}^N)$ *if and only if*

$$\liminf_{\delta\to 0_+} I_\delta(g) < +\infty.$$

*Moreover,*

$$\lim_{\delta\to 0_+} I_\delta(g) = \frac{1}{p} K_{N,p} \int_{\mathbb{R}^N} |Dg|^p \, dx, \quad \forall g \in W^{1,p}(\mathbb{R}^N),$$

*where* $K_{N,p}$ *is given by*

$$K_{N,p} = \int_{\mathbb{S}^{N-1}} |e \cdot \sigma|^p \, d\sigma,$$

*for any* $e \in \mathbb{S}^{N-1}$.

We recall that when $p = 1$,

(a) If $g \in L^1(\mathbb{R}^N)$ and $\liminf_{\delta\to 0_+} I_\delta(g) < +\infty$, then $g \in BV(\mathbb{R}^N)$ (see [3,12]).
(b) $\exists g \in W^{1,1}(\mathbb{R})$ such that $\lim_{\delta\to 0_+} I_\delta(g) = +\infty$ (example communicated to us by A. Ponce; see [10]).

This characterization is distinct from the one of J. Bourgain, H. Brezis, and P. Mironescu [1] (see also [5]) but it is inspired by the results of [1]. Quantities similar to $I_\delta$ appear in new estimates for the degree (see [2,11,6]). Further results related to Theorem 1 are presented in [12,14] and in a recent work of D. Chiron [7].

Let $p \geqslant 1$ and $N \geqslant 1$. Define, for $g \in L^p(\mathbb{R}^N)$,

$$J(g) = \begin{cases} \frac{1}{p} K_{N,p} \int_{\mathbb{R}^N} |Dg|^p \, dx & \text{if } p > 1 \text{ and } g \in W^{1,p}(\mathbb{R}^N) \text{ (resp. } p = 1 \text{ and } g \in BV(\mathbb{R}^N)), \\ +\infty & \text{otherwise.} \end{cases}$$

A natural question raised by H. Brezis (personal communication) is whether $(I_\delta)$ $\Gamma$-converges in $L^p(\mathbb{R}^N)$ to $J$ in the sense of E. De Giorgi for $p > 1$ (see e.g. [4,9] for an introduction of $\Gamma$-convergence). We recall that a family $(I_\delta)_{\delta\in(0,1)}$ of functionals defined on $L^p(\mathbb{R}^N)$ $\Gamma$-converges in $L^p(\mathbb{R}^N)$, as $\delta$ goes to 0, to a functional $I$ defined on $L^p(\mathbb{R}^N)$ if and only if the following two conditions are satisfied:

(G1) For each $g \in L^p(\mathbb{R}^N)$ and for every family $(g_\delta)_{\delta\in(0,1)} \subset L^p(\mathbb{R}^N)$ such that $g_\delta$ converges to $g$ in $L^p(\mathbb{R}^N)$ as $\delta$ goes to 0, one has

$$\liminf_{\delta\to 0} I_\delta(g_\delta) \geqslant I(g).$$

(G2) For each $g \in L^p(\mathbb{R}^N)$, there exists a family $(g_\delta)_{\delta\in(0,1)} \subset L^p(\mathbb{R}^N)$ such that $g_\delta$ converges to $g$ in $L^p(\mathbb{R}^N)$ as $\delta$ goes to 0, and

$$\limsup_{\delta\to 0} I_\delta(g_\delta) \leqslant I(g).$$

Surprisingly, $(I_\delta)$ does not $\Gamma$-converge to $J$ in $L^p(\mathbb{R}^N)$ for $p > 1$ but it $\Gamma$-converges to $\lambda J$ for some $0 < \lambda < 1$; the same fact holds for the case $p = 1$. More precisely, we have

**Theorem 2.** *Let $p \geqslant 1$ and $N \geqslant 1$. Then $(I_\delta)$ $\Gamma$-converges in $L^p(\mathbb{R}^N)$ to $I$ defined by, for all $g \in L^p(\mathbb{R}^N)$,*

$$I(g) = \begin{cases} C_{N,p} \int_{\mathbb{R}^N} |Dg|^p \, dx & \text{if } p > 1 \text{ and } g \in W^{1,p}(\mathbb{R}^N) \text{ (resp. } p = 1 \text{ and } g \in BV(\mathbb{R}^N)), \\ +\infty & \text{otherwise.} \end{cases}$$

*Here the constant $C_{N,p}$ is defined by* (2) *below and satisfies*

$$0 < C_{N,p} < \frac{1}{p} K_{N,p}. \tag{1}$$

For $p \geqslant 1$ and $N \geqslant 1$, the definition of the constant $C_{N,p}$ is the following

$$C_{N,p} := \inf \liminf_{\delta\to 0} \iint_{\substack{Q^2 \\ |h_\delta(x)-h_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy, \tag{2}$$

where the infimum is taken over all families of measurable functions $(h_\delta)_{\delta\in(0,1)}$ defined on the unit open cube $Q$ of $\mathbb{R}^N$ such that $h_\delta$ converges to $h(x) \equiv \frac{x_1+\cdots+x_N}{\sqrt{N}}$ in (Lebesgue) measure on $Q$ as $\delta$ goes to 0.

## 2. Sketch of the proof

The proof is quite long (about 40 pages) and it is divided into three steps:
*Step 1*: Proof of Property (G2).

**Claim 1.** *Let $p \geqslant 1$ and $N \geqslant 1$. Then for each $g \in W^{1,p}(\mathbb{R}^N)$ if $p > 1$, or $g \in BV(\mathbb{R}^N)$ if $p = 1$, there exists a family $(g_\delta)_{\delta\in(0,1)} \subset L^p(\mathbb{R}^N)$ such that $g_\delta$ converges to $g$ in $L^p(\mathbb{R}^N)$ as $\delta$ goes to 0, and*

$$\limsup_{\delta\to 0} I_\delta(g_\delta) \leqslant I(g).$$

The proof of Claim 1 is quite involved. We mention here main steps of the proof for the case $N = 1$:

(a) We show that there exists a family $(h_\delta)$ in $L^p(0,1)$ defined for all $\delta \in (0,1)$ (not just for a sequence $\delta_n \to 0$) such that $h_\delta$ converges to $h(x) \equiv x$ in $L^p(0,1)$ and

$$\lim_{\delta\to 0} \iint_{\substack{[0,1]^2\\ |h_\delta(x)-h_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy = C_{1,p}.$$

(b) We prove Claim 1 in the case $g$ is continuous and piecewise linear with compact support. To this end, on each interval $K$ where $g$ is linear, using (a) we can find a family of functions $(h_{K,\delta}) \subset L^p(K)$ such that $h_{K,\delta}$ converges to $g$ in $L^p(K)$ and

$$\lim_{\delta\to 0} \iint_{\substack{K^2\\ |h_{K,\delta}(x)-h_{K,\delta}(y)|>\delta}} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy = C_{1,p}\big|g'(x_0)\big|^p|K|,$$

for some $x_0 \in K$. Then we glue these functions and construct a function $g_\delta$ on $\mathbb{R}$. This is delicate since $I_\delta$ is very sensitive to jumps.

(c) We deduce Claim 1 from (b) by using the fact that if $g$ is as in Claim 1, then there exists a sequence of continuous and piecewise linear functions with compact support $(\phi_n)$ such that $\phi_n$ converges to $g$ in $L^p(\mathbb{R})$ and $\|D\phi_n\|_{L^p(\mathbb{R})}$ converges to $\|Dg\|_{L^p(\mathbb{R})}$ (when $p = 1$ the $L^1$-norm is replaced by the total mass).

Proof of Property (G2) follows from Claim 1 and the definition of $I$.

*Step 2*: Proof of Property (G1).

**Claim 2.** *Let $p \geqslant 1$ and $N \geqslant 1$. Then for any $g \in W^{1,p}(\mathbb{R}^N)$ if $p > 1$ or $g \in BV(\mathbb{R}^N)$ if $p = 1$, and for any family $(g_\delta)_{\delta\in(0,1)} \subset L^p(\mathbb{R}^N)$ such that $g_\delta$ converges to $g$ in $L^p(\mathbb{R}^N)$ as $\delta$ goes to 0, one has*

$$\liminf_{\delta\to 0} I_\delta(g_\delta) \geqslant I(g).$$

The proof of Claim 2 for the case $p > 1$ and $N = 1$ follows from the definition of $C_{1,p}$ and the fact that any function in $W^{1,p}(\mathbb{R})$ is locally approximately linear in the sense of measure (see e.g. [8, Theorem 4 on page 223] and the remark below it). In the case $p > 1$ and $N > 1$, one uses the same idea as in the one dimensional case. However, it is more technical. When $p = 1$, we can not directly apply the method used in the case $p > 1$. In this case, the proof relies on some new ingredients and a new characterization of BV functions which we introduce in [13]. In the proof, we also use the structure theorem for BV functions (see e.g. [8, Theorem 1 on page 167]), the differentiation theorem of Radon measures (see e.g. [8, Theorem 1 on page 38]) and Besicovitch's covering theorem.

**Claim 3.** *Let $p \geqslant 1$, $N \geqslant 1$, and $g \in L^p(\mathbb{R}^N)$. Assume that there exists a family $(g_\delta)_{\delta\in(0,1)} \subset L^p(\mathbb{R}^N)$ such that $g_\delta$ converges to $g$ in $L^p(\mathbb{R}^N)$ and*

$$\liminf_{\delta\to 0} I_\delta(g_\delta) < +\infty.$$

*Then $g \in W^{1,p}(\mathbb{R}^N)$ if $p > 1$ (resp. $g \in BV(\mathbb{R}^N)$ if $p = 1$); moreover*

$$J(g) \leqslant C \liminf_{\delta\to 0} I_\delta(g_\delta),$$

*for some $C > 0$ depending only on $N$ and $p$.*

Claim 3 was proved in [12] (see [12, Theorem 3]); the proof in [12] relies heavily on the ideas of [3]. Property (G1) now follows from Claims 2 and 3.

*Step 3*: Proof of (1).

Let $g$ and $g_\delta$ be defined on $\mathbb{R}^N$ by

$$g(x) = \begin{cases} |x| & \text{if } |x| \leqslant 1,\\ 1/|x|^{2N} & \text{otherwise,}\end{cases}$$

and

$$g_\delta(x) = \begin{cases} (k+1)\delta & \text{if } k\delta \leqslant |x| < (k+1)\delta \text{ for } 0 \leqslant k \leqslant [1/\delta], \\ 1/|x|^{2N} & \text{otherwise.} \end{cases}$$

Here $[1/\delta]$ denotes the largest integer less than $1/\delta$. Then $g_\delta$ converges to $g$ in $L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ as $\delta$ goes to 0. On the other hand, it is easy to see that

$$\liminf_{\delta\to 0}\big[I_\delta(g) - I_\delta(g_\delta)\big] \geqslant \liminf_{\delta\to 0} \sum_{k=0}^{[1/\delta]} \iint\limits_{\substack{k\delta \leqslant |x| \leqslant (k+\frac{1}{2})\delta \\ (k+\frac{3}{2})\delta \leqslant |y| \in (k+2)\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy,$$

$$\lim_{\delta\to 0} I_\delta(g) = \frac{1}{p} K_{N,p} \int\limits_{\mathbb{R}^N} |Dg|^p\,dx,$$

and

$$\liminf_{\delta\to 0} \sum_{k=0}^{[1/\delta]} \iint\limits_{\substack{k\delta \leqslant |x| \leqslant (k+\frac{1}{2})\delta \\ (k+\frac{3}{2})\delta \leqslant |y| \leqslant (k+2)\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy > 0.$$

As a result, one obtains

$$\limsup_{\delta\to 0} I_\delta(g_\delta) < \frac{1}{p} K_{N,p} \int\limits_{\mathbb{R}^N} |Dg|^p\,dx,$$

and therefore,

$$C_{N,p} < \frac{1}{p} K_{N,p}.$$

On the other hand, as a consequence of Claims 1 and 3, one has $C_{N,p} > 0$. This completes the proof of Step 3.

We do not know the explicit value of the constant $C_{N,p}$. But we have a guest when $N = 1$. Let $h$ and $h_n$ be functions defined on (0, 1) by $h(x) = x$ on (0, 1) and $h_n(x) = \frac{k-1}{n}$ if $\frac{k-1}{n} \leqslant x < \frac{k}{n}$ for $1 \leqslant k \leqslant n$. An easy computation shows that

$$\lim_{n\to\infty} \iint\limits_{\substack{[0,1]^2 \\ |h_n(x)-h_n(y)|>1/n}} \frac{1/n^p}{|x-y|^{p+1}}\,dx\,dy = c_{1,p},$$

where

$$c_{1,p} = \begin{cases} \frac{2}{p(p-1)}\left(1 - \frac{1}{2^{p-1}}\right) & \text{if } p > 1, \\ 2\ln 2 & \text{if } p = 1. \end{cases}$$

Clearly,

$$c_{1,p} \geqslant C_{1,p}.$$

The following open question is suggested:

**Open question 1.** Is $C_{1,p}$ equal to $c_{1,p}$?

The detailed proofs of the results discussed in this Note will be presented in [13].

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