## Differential Geometry

# The constant osculating rank of the Wilking manifold $V_{3}{ }^{\text {** }}$ 

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#### Abstract

We prove that the osculating rank of the Wilking manifold $V_{3}=(S O(3) \times S U(3)) / U^{\bullet}(2)$, endowed with the metric $\tilde{g}_{1}$, equals 2 . The knowledge of the osculating rank allows us to solve the differential equation of the Jacobi vector fields. These results can be applied to determine the area and the volume of geodesic spheres and balls. To cite this article: E. Macías-Virgós et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008). © 2007 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.


## Résumé

Le rang osculateur constant de la variété de Wilking $\boldsymbol{V}_{\mathbf{3}}$. Nous prouvons que le rang osculateur de la variété de Wilking $V_{3}=(S O(3) \times S U(3)) / U^{\bullet}(2)$ vaut 2 , lorsque on considère la metrique $\tilde{g}_{1}$. La connaissance du rang osculateur nous permet de resoudre l'équation différentielle des champs de vecteurs de Jacobi. Ces résultats peuvent être appliqués pour déterminer l'aire et le volume des sphères et boules géodesiques. Pour citer cet article: E. Macías-Virgós et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).
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## 1. Introduction

Solving the Jacobi equation on a Riemannian manifold can be a rather difficult task. For the manifolds $V_{1}=$ $S p(2) / S U(2)$ and $V_{2}=S U(5) / S p(2) \times S^{1}$ a partial solution was obtained by I. Chavel in [5,6]. It is well known that these manifolds are non-symmetric normal homogeneous spaces of rank 1 [3, p. 237]. A theorem from Berger [3] establishes that a simply connected, normal homogeneous space of positive sectional curvature is diffeomorphic either to a compact rank-one symmetric space or to one of the manifolds $V_{1}$ or $V_{2}$. In [15] Wilking proves that this theorem is not correct because there is a third exception, the manifold $V_{3}=(S O(3) \times S U(3)) / U^{\bullet}(2)$, equipped with a one-parameter family $\tilde{g}_{\lambda}$ of bi-invariant metrics.

[^0]In [14] Tsukada concludes that the Jacobi tensor field over an n.r.h.s. (naturally reductive homogeneous space, see the definition in [6], [10, p. 202]) has constant osculating rank $r \in \mathbb{N}$. Therefore, for the Jacobi operator one has

$$
R_{t}=R_{0}+a_{1}(t) R_{0}^{1)}+\cdots+a_{r}(t) R_{0}^{r)}
$$

T. Arias-Marco and the second author have extended this interesting geometric result to g.o. spaces [1]. In [11], two of the authors, always working with the Levi-Civita connection and using the result of Tsukada, have proved that the manifold $V_{1}$ has constant osculating rank 2. Also, they solved the Jacobi equation along a unit geodesic of arbitrary direction for this manifold. The solutions are applied to obtain the area of the geodesic spheres and the volume of the geodesic balls.

Given its generality, we have thought that this method could also be applied to solve the Jacobi equation in several other examples of n.r.h.s. In this Note we are going to do the same study for the Wilking's manifold ( $V_{3}, \tilde{g}_{\lambda}$ ). We prove that it has osculating rank 2 when $\lambda=1$. This result allows us to solve the Jacobi equation in such case.

## 2. Preliminaries

Let $M$ be an $n$-dimensional, connected, real analytical Riemannian manifold, $m \in M$. Let $\gamma: J \rightarrow M$ be a geodesic defined on some open interval $J \subset \mathbb{R}$ with $0 \in J, m=\gamma(0)$. The associated Jacobi operator $R_{t}$ is the self-adjoint tensor field along $\gamma=\gamma(t)$ defined by $R_{t}(\cdot):=R\left(\cdot, \gamma^{\prime}\right) \gamma^{\prime}(t)$. For the curvature tensor we follow the notations of [10]. The covariant derivative $R_{t}^{i)}$ of the Jacobi operator $R_{t}$ along $\gamma$ is the self-adjoint tensor field defined by

$$
R_{t}^{i)}(\cdot):=\left(\nabla_{\gamma^{\prime}}{ }^{i)} \cdots \nabla_{\gamma^{\prime}} R\right)\left(\cdot, \gamma^{\prime}\right) \gamma^{\prime}(t),
$$

where $\nabla$ is the Levi-Civita connection associated to the metric. Its value at $\gamma(0)$ will be denoted by $R_{0}^{i)}(\cdot)$.

Theorem 1. (See [11].) For $n \geqslant 1$ we have

$$
\nabla_{\gamma^{\prime}}{ }^{n)} \nabla_{\gamma^{\prime}} R\left(X, \gamma^{\prime}\right) \gamma^{\prime}=\sum_{i=0}^{n}\binom{n}{i} R_{t}^{n-i)}\left(\nabla_{\gamma^{\prime}}{ }^{i)} \nabla_{\gamma^{\prime}} X\right)
$$

Let $G$ be a Lie group, $H$ a closed subgroup. The Lie algebras of $G$ and $H$ will be denoted by $\mathfrak{g}$ and $\mathfrak{h}$, respectively. We identify the vector space $\mathbf{m}=\mathfrak{g} / \mathfrak{h}$ with the tangent space to $G / H$ at the origin $o$.

From now on we shall assume that $G / H$ is an n.r.h.s. If we define $\Lambda: \mathbf{m} \times \mathbf{m} \rightarrow \mathbf{m}$ by $\Lambda(u) v=(1 / 2)[u, v]_{\mathbf{m}}$ for $u, v \in \mathbf{m}$, we can identify $\Lambda$ with the Levi-Civita connection $\nabla$ [10, Ch. X, p. 201]. Accordingly to [13] and [10, Vol. II, Ch. X], for each $v \in \mathbf{m}$ the (1,3)-tensor $R_{t}$ on $\mathbf{m}$ obtained by the parallel translation of the Jacobi operator along $\gamma$ is given as

$$
R_{t}=\tau(\exp t v)_{*} \cdot e^{-t \Lambda(v)} \cdot R_{0}
$$

where $R_{0}$ denotes the Jacobi operator at the origin $o$.
Proposition 2. (See [6],[10, Vol. II, p. 202].) Let $\gamma(t)$ be a geodesic with $\gamma(0)=o, \gamma^{\prime}(0)=v \in \mathbf{m}$. If $X$ is a differentiable vector field along $\gamma$, then

$$
R_{0}(X)=-\left[[X, v]_{\mathbf{h}}, v\right]-(1 / 4)\left[[X, v]_{\mathbf{m}}, v\right]_{\mathbf{m}} .
$$

Proposition 3. (See [11].) For $n>0$,

$$
\begin{equation*}
(-1)^{n-1} 2^{n} R_{0}^{n)}(X)=\sum_{i=0}^{n}(-1)^{i}\binom{n}{i}\left[\left[[X, v]_{\mathbf{m}}, \ldots, v\right]_{\mathbf{h}}^{i+1)}, \ldots, v\right]_{\mathbf{m}}, \tag{1}
\end{equation*}
$$

where for each term of the sum we have $n+2$ brackets and the exponent $i+1$ ) means the position of the bracket valued on $\mathbf{h}$.

## 3. An explicit form for the Jacobi operator

In [15] the author proves that Berger's classification is not totally correct. In fact, he shows that there is a third example $V_{3}=(S O(3) \times S U(3)) / U^{\bullet}(2)$ equipped with a one-parameter family $\tilde{g}_{\lambda}$ of bi-invariant metrics, where $U^{\bullet}(2)$ is the image under the embedding $(\pi, i): U(2) \rightarrow S O(3) \times S U(3)$ given by the natural inclusion $i(A)=\operatorname{diag}\left(A, \operatorname{det} A^{-1}\right)$ and the projection $\pi: U(2) \rightarrow U(2) / S^{1}$. Using the natural isomorphism between $\mathfrak{s o}(3)$ and $\mathfrak{s u}(2)$, we can consider the Lie algebra $\mathfrak{s o}(3) \oplus \mathfrak{s u}(3)$ of $S O(3) \times S U(3)$ as the subalgebra of $\mathfrak{s u} u(5)$ of matrices of the form $X=\operatorname{diag}\left(X_{1}, X_{2}\right)$ with $X_{1} \in \mathfrak{s u} u(2), X_{2} \in \mathfrak{s u}(3)$. Then an element of the Lie algebra $\mathfrak{u}^{\bullet}(2)$ of $U^{\bullet}(2)$ may be expressed as a diagonal block matrix $\operatorname{diag}\left(\pi_{*} A, A,-\operatorname{trace} A\right)$ with $A \in \mathfrak{u}(2)$, where $\pi_{*}$ denotes the differential of $\pi$. We shall consider the bi-invariant metrics $\langle,\rangle_{c}, c>0$, on $S O(3) \times S U(3)$ given by

$$
\langle X, Y\rangle_{c}=(-1 / 2)\left(c \text { trace } X_{1} Y_{1}+\operatorname{trace} X_{2} Y_{2}\right), \quad X, Y \in \mathfrak{s} o(3) \oplus \mathfrak{s} u(3)
$$

They induce a one parametric family of metrics on $V_{3}$. The corresponding Wilking's metrics are $\tilde{g}_{\lambda}=12\langle,\rangle_{c}$, $c=3 \lambda / 2$. Then $\left(V_{3}, \tilde{g}_{\lambda}\right)$ is isometric to the Aloff-Wallach space $\left(M_{11}^{7}, g_{t}\right)$ [2], for $t=-3 /(2 \lambda+3)$.

Next, we choose an orthonormal basis of $\mathfrak{s o}(3) \oplus \mathfrak{s} u(3)$ adapted to the reductive decomposition $\mathfrak{u} \bullet(2) \oplus \mathbf{m}$, see also [7]. For $1 \leqslant k, l \leqslant 5$, let $E_{k l}$ denote the $5 \times 5$ matrices $\left(\delta_{a k} \delta_{b l}\right)_{1 \leqslant a, b \leqslant 5 \text {. We introduce the matrices }}$

$$
Q_{k l}=E_{k l}-E_{l k}, \quad R_{k l}=\sqrt{-1}\left(E_{k l}+E_{l k}\right), \quad P_{k}=\sqrt{-1}\left(E_{k k}+E_{55}\right)
$$

An orthonormal basis $\left\{M_{1}, \ldots, M_{7}\right\}$ for $\mathbf{m}$ is given by

$$
\begin{array}{ll}
\left.M_{1}=1 / \sqrt{c(1+c}\right)\left(P_{1}-P_{2}-c\left(P_{3}-P_{4}\right)\right), & M_{4}=Q_{35}, \quad M_{7}=R_{45} \\
M_{2}=1 / \sqrt{c(1+c)}\left(R_{12}-c R_{34}\right), & M_{5}=Q_{45} \\
M_{3}=1 / \sqrt{c(1+c)}\left(Q_{12}-c Q_{34}\right), & M_{6}=R_{35}
\end{array}
$$

while a basis for $\mathfrak{u}^{\bullet}(2)$ is given by

$$
\begin{array}{ll}
M_{8}=1 / \sqrt{1+c}\left(Q_{12}+Q_{34}\right), & M_{10}=1 / \sqrt{1+c}\left(P_{1}-P_{2}+P_{3}-P_{4}\right) \\
M_{9}=1 / \sqrt{1+c}\left(R_{12}+R_{34}\right), & M_{11}=1 / \sqrt{3}\left(P_{3}+P_{4}\right)
\end{array}
$$

Now it is necessary to compute the matrices of the Jacobi operator $R_{0}$ and its derivatives $R_{0}^{1)}, R_{0}^{2)}, R_{0}^{3)}, \ldots$ in order to know if there exists any dependence relation among them.

Lemma 4. For the metric $\tilde{g}_{1}$ (i.e. $c=2 / 3$ ), at $\gamma(0)$ we have:
(i) $5 R_{0}^{3)}=-2\left\|\gamma^{\prime}\right\|^{2} R_{0}^{1)}=-2 R_{0}^{1)}$;
(ii) $5 R_{0}^{4)}=-2\left\|\gamma^{\prime}\right\|^{2} R_{0}^{2)}=-2 R_{0}^{2)}$.

In order to prove the lemma it is necessary to compute all the brackets between the vectors of the basis $\left\{M_{1}, \ldots, M_{11}\right\}$ and to use the formula given in Proposition 3. The results and an explicit computation with Mathematica can be seen at the electronic address http://xtsunxet.usc.es/macias/wilkingV3.

Remark 1. In fact we have computed the derivatives $R_{0}^{1)}$ and $R_{0}^{3)}$ for any value of $\lambda$ (see the website above). These matrices are tensors on the coordinates $x_{1}, \ldots, x_{7}$ in $\mathbf{m}$. When computing $R^{5)}$ we found that the product $x^{1} \cdots x^{7}$ appears with a coefficient which is null if and only if $\lambda=1$. Hence, in general it is not possible to write a dependence relation like

$$
R^{5)}=a\|\gamma\|^{2} R^{3)}+b\|\gamma\|^{4} R^{1)}
$$

where $\|\gamma\|^{2}=x_{1}^{2}+\cdots+x_{7}^{2}=1$ is the module of the geodesic. This obstruction suggests that the osculating rank is greater than 5 for any of the metrics $\tilde{g}_{\lambda}$, with $\lambda \neq 1$.

Remark 2. The metric $\tilde{g}_{1}$ is the natural one induced by the Killing form on $\mathfrak{s o}(3) \times \mathfrak{s u}(3)$. It is an Einstein metric and gives the optimal pinching constant among the positively curved Aloff-Wallach metrics on $M_{11}^{7}$ [12].

Now, in a similar way that in [11], by using the induction method we have for the metric $\tilde{g}_{1}$ :
Proposition 5. At $\gamma(0)$ we have:
(i) $R_{0}^{2 n)}=(-1)^{n-1}(2 / 5)^{n-1} R_{0}^{2)}$;
(ii) $R_{0}^{2 n+1)}=(-1)^{n}(2 / 5)^{n} R_{0}^{1)}$.

Corollary 6. Along the geodesic $\gamma$ the Jacobi operator can be written as

$$
R_{t}=R_{0}+\frac{5}{2} R_{0}^{2)}+\frac{\sin (\sqrt{2 / 5} t)}{\sqrt{2 / 5}} R_{0}^{1)}-\frac{\cos (\sqrt{2 / 5} t)}{2 / 5} R_{0}^{2)}
$$

The proof follows from the Taylor's development of $R_{t}$ at $t=0$.
Remark 3. The solution of the Jacobi equation on the manifold $V_{3}$ allows us to determine the volume of geodesic spheres and balls in this manifold. We use for the computations the standard methods given in [4,8,9]. An analogous study was done for $V_{1}$ in [11].

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