# Multiscale method based on discontinuous Galerkin methods for homogenization problems 

Assyr Abdulle<br>School of Mathematics and Maxwell Institute for Mathematical Sciences, University of Edinburgh, Edinburgh EH9 3JZ, UK

Received 24 October 2007; accepted after revision 22 November 2007
Available online 31 December 2007
Presented by Philippe G. Ciarlet


#### Abstract

We propose a multiscale method for elliptic problems with highly oscillating coefficients based on a coupling of macro and micro methods in the framework of the heterogeneous multiscale method. The macro method, defined on a macroscopic triangulation, aims at recovering the effective (homogenized) solution of an unknown macro model. The unspecified data of this model are computed by micro methods on sampling domains during the macro assembly process. In this Note, we show how to construct such a coupling with a discontinuous macro finite element space. We show that the flux information needed in this formulation in order to impose weak interelement continuity can be recovered from the known micro calculations on the sampling domains. A fully discrete analysis is presented. To cite this article: A. Abdulle, C. R. Acad. Sci. Paris, Ser. I 346 (2008). © 2007 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.


## Résumé

Schéma multi-échelles basé sur une méthode de Galerkin discontinue pour des problèmes d'homogénéisation. Nous proposons une méthode multi-échelles combinant un schéma macroscopique et des schémas microscopiques pour la résolution numérique d'équations elliptiques avec des coefficients fortement oscillants. Le schéma macroscopique, basé sur un macro-maillage, a pour objectif de fournir une approximation du problème effectif (homogénéisé). Les paramètres de ce schéma, a priori inconnus, sont obtenus pendant l'assemblage du problème effectif, à l'aide de schémas microscopiques mis en oeuvre sur des micro-cellules contenues dans le macro-maillage. Dans cette Note, nous expliquons comment ce couplage peut-être réalisé avec un schéma macroscopique basé sur une méthode de Galerkin discontinue. Nous montrons que les flux locaux nécessaire à la mise en oeuvre d'une telle méthode peuvent être construits à l'aide des solutions disponibles dans les cellules microscopiques. Une analyse d'erreur globale des schémas couplés est présentée. Pour citer cet article : A. Abdulle, C. R. Acad. Sci. Paris, Ser. I 346 (2008). © 2007 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

## Version française abrégée

Le développement de méthodes multi-échelles pour la résolution d'équations aux dérivées partielles (EDP) avec des coefficients fortement oscillants est essentiel pour de nombreuses applications. Récemment, de nouvelles méthodes de type macro-micro ont été proposées pour la résolution numérique de ce type d'EPD. Ces méthodes sont

[^0]construites à l'aide d'un couplage entre un schéma macroscopique mis en oeuvre sur une équation effective a priori inconnue, et des schémas microscopiques mis en oeuvre sur des micro-cellules et fournissant, pendant le processus d'assemblage, les paramètres de l'équation effective [ $2-4,10,11]$. Le maillage macroscopique de telles méthodes combiné avec une résolution de la structure fine sur des cellules de petite taille, entraîne une réduction significative des coûts comparés à ceux obtenus avec des méthodes d'élément finis classiques pour le type d'EDP considérées. Dans cette Note, nous proposons un nouveau type de couplage avec un schéma macroscopique basé sur une méthode de Galerkin discontinue [5,6]. Il est connu que ce type de méthodes d'élément finis, basés sur une formulation locale possèdent de nombreuses propriétés favorables (conservation locale de la masse, flexibilité dans le maillage, adaptivité $[5,6]$ ). La solution numérique obtenue avec de telles méthodes peut être discontinue à la frontière des éléments finis. Ces discontinuités doivent être controlées par l'introduction de flux locaux entre les éléments. Pour des EDP avec des coefficients fortement oscillants et dans le contexte d'une méthode multi-échelles nous montrons que la construction de flux macroscopiques peut être obtenue avec l'information disponible dans les micro-cellules mentionnées ci-dessus. Une analyse de la méthode multi-échelles tenant compte de l'erreur introduite par les différents maillages est proposée.

## 1. Introduction

We consider the second-order elliptic problem in the convex domain $\Omega \subset \mathbb{R}^{d}$

$$
\begin{equation*}
-\nabla \cdot\left(a^{\varepsilon} \nabla u^{\varepsilon}\right)=f \quad \text { in } \Omega, \quad u^{\varepsilon}=0 \quad \text { on } \partial \Omega, \tag{1}
\end{equation*}
$$

where $a^{\varepsilon}$ is symmetric, satisfy $a^{\varepsilon}(x) \in\left(L^{\infty}(\Omega)\right)^{d \times d}$ and is uniformly elliptic and bounded, i.e.,

$$
\begin{equation*}
\exists \lambda, \Lambda>0 \text { such that } \lambda|\xi|^{2} \leqslant a^{\varepsilon}(x) \xi \cdot \xi \leqslant \Lambda|\xi|^{2}, \forall \xi \in \mathbb{R}^{d} \text { and } \forall \varepsilon \text {, } \tag{2}
\end{equation*}
$$

where $\varepsilon$ represent a small scale in the problem that characterize the multiscale nature of the tensor $a^{\varepsilon}(x)$ (i.e., by varying $\varepsilon$ we consider a family of tensor $\left\{a^{\varepsilon}\right\}$ with the above properties). We further assume that $f \in L^{2}(\Omega)$. An application of Lax-Milgram theorem gives us a family of solutions $\left\{u^{\varepsilon}\right\}$ which are bounded in $H_{0}^{1}(\Omega)$. Without further assumptions on the heterogeneities of the tensor $a^{\varepsilon}(x)$, using the notion of $G$-convergence introduced by De Giorgi and Spagnolo [9], one can show that there exists a symmetric tensor $a^{0}(x)$ and a subsequence of $\left\{u^{\varepsilon}\right\}$ which weakly converges to an element $u^{0} \in H_{0}^{1}(\Omega)$ solution of the so-called homogenized or upscaled problem

$$
\begin{equation*}
-\nabla \cdot\left(a^{0} \nabla u^{0}\right)=f \quad \text { in } \Omega, \quad u^{0}=0 \quad \text { on } \partial \Omega, \tag{3}
\end{equation*}
$$

where the homogenized tensor $a^{0}(x)$ again satisfies $\lambda|\xi|^{2} \leqslant a^{0}(x) \xi \cdot \xi \leqslant \Lambda|\xi|^{2}, \forall \xi \in \mathbb{R}^{d}$.
In what follows, we construct a multiscale finite element method (FEM) to approximate the solution $u_{0}$ of the homogenized problem which does not rely on precomputing the homogenized tensor $a^{0}$. The primary goal of a method as the heterogeneous multiscale method (HMM) is to compute the macro state (here $u_{0}$ ) of a multiscale problem. In this Note, we construct a macro-to-micro method which uses a macro solver based on a discontinuous Galerkin (DG) FEM. This allows to benefit from many advantageous features of such methods (local conservation properties, flexibility in approximation and meshing [6]) for the macro dynamics. It permits to use nonmatching meshes, to triangulate complicated domains, and allows for local mass or flux conservation for the macro dynamics. Furthermore, for time-dependent problems, such multiscale methods give a bloc-diagonal macro mass matrix and they can thus easily be combined with explicit stabilized solver in time. Since the coupling of micro and macro methods is inherently local in the HMM, a coupling with a macro DG solver may also be efficient for local adaptivity and parallel implementation.

A DG multiscale method based on HMM has been proposed in [8] for one-dimensional hyperbolic and parabolic problems and a DG-FEM for elliptic problem has recently been proposed for homogenization problems (in a different framework as the one developed below) but not analyzed [1]. To the best of our knowledge, this is the first analysis of a multiscale DG-FEM for elliptic homogenization problems.

## 2. Discontinuous Galerkin (DG) method

In this section we briefly recall some notation for finite element method (FEM) based on discontinuous finite element (FE) space. For simplicity we consider piecewise linear FE spaces and we suppose that $\Omega$ is a convex polygon. For DG-FEM we relax the standard interelement continuity for FEM and define

$$
\begin{equation*}
V_{h} \equiv V_{h}\left(\Omega, \mathcal{T}_{h}\right)=\left\{u^{h} \in L^{2}(\Omega) ;\left.u^{h}\right|_{K} \in \mathcal{P}^{1}(K), \forall K \in \mathcal{T}_{h}\right\} \tag{4}
\end{equation*}
$$

where $\mathcal{T}_{h}$ is a shape regular triangulation of $\Omega, \mathcal{P}^{1}(K)$ is the space of linear polynomials on the triangle $K$ and $h=$ $\max h_{K}, K \in \mathcal{T}_{h}\left(h_{K}\right.$ is the size of $\left.K\right)$. We also define the piecewise Sobolev space $H^{l}\left(\mathcal{T}_{h}\right):=\prod_{K \in \mathcal{T}_{h}} H^{l}(K)=\{v \in$ $\left.L^{2}(\Omega) ;\left.v\right|_{K} \in H^{l}(K) \forall K \in \mathcal{T}_{h}\right\}$. We note that $V_{h} \subset H^{1}\left(\mathcal{T}_{h}\right)$. In what follows, we focus on a specific DG formulation, that is, the interior penalty DG-FEM (see [6]), but we note that our multiscale method could be adapted to other formulations. We consider an arbitrary element $K$ of our triangulation $\mathcal{T}_{h}$, multiply the problem (1) with a smooth test function $v$ and integrate by parts using $a^{\varepsilon} \nabla u^{\varepsilon} \in H$ (div, $K$ ). Summing over $K \in \mathcal{T}_{h}$ yields

$$
\begin{equation*}
\int_{\Omega} a^{\varepsilon} \nabla u^{\varepsilon} \cdot \nabla v \mathrm{~d} x-\sum_{K \in \mathcal{I}_{h \partial K}} \int_{a^{\varepsilon}} a^{\varepsilon} \nabla u^{\varepsilon} \cdot n_{K} v \mathrm{~d} s=\int_{\Omega} f v \mathrm{~d} x \tag{5}
\end{equation*}
$$

where $n_{K}$ is the outward normal. We denote by $e \in \mathcal{E}$ an interface shared by two neighboring elements $K_{1}$ and $K_{2}$, where $\mathcal{E}$ is the set of all (interior and boundary) interfaces. Since hanging nodes are allowed, $\mathcal{E}$ will be understood to contain the smallest common interfaces of neighboring elements. For a piecewise smooth function $\xi$ (possibly vector valued) we denote by $\xi_{1}, \xi_{2}$ its trace from within $K_{1}, K_{2}$, respectively, and consider the jump and the average defined by $\{\xi\}=\frac{1}{2}\left(\xi_{1}+\xi_{2}\right), \llbracket \xi \rrbracket=\xi_{1} n_{1}+\xi_{2} n_{2}$, where $n_{i}$ denotes the unit outward normal vectors on the interface $K_{i}$. Notice that $\llbracket \xi \rrbracket$ is a vector-valued function if $\xi$ is a scalar function, while it is a scalar function if $\xi$ is a vector-valued function. Using these notations we can rewrite (5) as

$$
\begin{equation*}
\int_{\Omega} a^{\varepsilon} \nabla u^{\varepsilon} \cdot \nabla v \mathrm{~d} x-\sum_{e \in \mathcal{E}} \int_{e}\left\{a^{\varepsilon} \nabla u^{\varepsilon}\right\} \llbracket v \rrbracket=\int_{\Omega} f v \mathrm{~d} x . \tag{6}
\end{equation*}
$$

Since the exact solution of (1) is in $H_{0}^{1}(\Omega)$ we have $\llbracket u^{\varepsilon} \rrbracket=0$ and we can make the bilinear form (6) symmetric by adding $-\sum_{e \in \mathcal{E}} \int_{e}\left\{a^{\varepsilon} \nabla v\right\} \llbracket u \rrbracket$ (assuming the existence of a trace for $a^{\varepsilon} \nabla v$ ). Finally to have a stable method one adds a penalty term. At the discrete level, we obtain the interior penalty DG-FEM (see [6]) for which one seeks a solution $u^{h} \in V_{h}$ such that

$$
\begin{align*}
& \int_{\Omega} a^{\varepsilon} \nabla u^{h} \cdot \nabla v^{h} \mathrm{~d} x-\sum_{e \in \mathcal{E}} \int_{e}\left(\left\{a^{\varepsilon} \nabla u^{h}\right\} \llbracket v^{h} \rrbracket+\left\{a^{\varepsilon} \nabla v^{h}\right\} \llbracket u^{h} \rrbracket\right) \mathrm{d} s+\sum_{e \in \mathcal{E}} \int_{e} \mu \llbracket u^{h} \rrbracket \llbracket u^{h} \rrbracket \\
& \quad=\int_{\Omega} f v^{h} \mathrm{~d} x \quad \forall v^{h} \in V_{h}, \tag{7}
\end{align*}
$$

where $\mu=\alpha h_{e}^{-1}$ with $\alpha>0$ independent of the meshsizes and $h_{e}$ is the interface size with the convention mentioned previously for hanging nodes. Here and in what follows, the gradient $\nabla$ should be understood as a broken gradient $\nabla_{h}$ when dealing with discontinuous functions, defined by $\left.\nabla_{h} u^{h}\right|_{K}=\nabla u, \forall K \in \mathcal{T}_{h}$. The choice of $\alpha$ is dictated by stability requirement. The analysis of this method as well as many other methods based on discontinuous Galerkin FE space is discussed in [6]. Several remarks are in order. First, it is well-known that for multiscale problems such as (1), $h<\varepsilon$ is required to have a good approximation and this is prohibitive in terms of computation costs if $\varepsilon$ is small. Second, regularity on $a^{\varepsilon}$ to be able to extend it up to $\partial K$ is needed and this may not be realistic for many problems with oscillating coefficients. In the method described below, we will only need to compute averages of quantities involving $a^{\varepsilon}$ on sampling domains and we will thus not require the existence of traces for $a^{\varepsilon}$.

## 3. A heterogeneous multiscale method (HMM) based on DG-FEM

Multiscale methods for homogenization problems based on HMM [10] have been developed in [2-4,11]. In this section we derive a DG-HMM based on the macro DG space $V_{H} \equiv V_{H}\left(\Omega, \mathcal{T}_{H}\right)$, i.e., the space (4) where $H$ is allowed to be much larger than $\varepsilon$. For a discretization in such a macro space, we need to modify the bilinear form in (7). In what follows, we consider within each macro triangle $K \in \mathcal{T}_{H}$ a sampling domain $K_{\delta} \subset K$.

Macro bilinear form. For $v^{H}, w^{H} \in V_{H}$ and each macro triangle $K \in \mathcal{T}_{H}$ we consider appropriate constrained micro functions $v^{h}, w^{h}$ on sampling domains $K_{\delta} \subset K$ and multiscale jumps $\left\{\overline{a^{\varepsilon} \nabla v^{h}}\right\},\left\{\overline{a^{\varepsilon} \nabla w^{h}}\right\}$ (see below). We then define a macro bilinear form on $V_{H} \times V_{H}$ by

$$
\begin{align*}
& B_{\mathrm{DG}}\left(u^{H}, v^{H}\right) \\
& \quad=\sum_{K \in \mathcal{T}_{H}} \omega_{K_{\delta}} \int_{K_{\delta}} a^{\varepsilon} \nabla u^{h} \cdot \nabla v^{h} \mathrm{~d} x-\sum_{e \in \mathcal{E}} \int_{e}\left(\left\{\overline{a^{\varepsilon} \nabla u^{h}}\right\} \llbracket v^{H} \rrbracket+\left\{\overline{a^{\varepsilon} \nabla v^{h}}\right\} \llbracket u^{H} \rrbracket\right) \mathrm{d} s+\sum_{e \in \mathcal{E}} \int_{e} \mu \llbracket u^{H} \rrbracket \llbracket v^{H} \rrbracket, \tag{8}
\end{align*}
$$

where $\mu$ is the discontinuity-penalization parameter defined by $\left.\mu\right|_{e}=\mu_{e}=\alpha H_{e}^{-1}$ (with the same convention as before for hanging nodes) and $\alpha$ is a positive parameter independent of the local meshsizes. The weight factors are chosen as $\omega_{K_{\delta}}=\frac{|K|}{\left|K_{\delta}\right|}$, where | | denotes the measure of the considered domains.

Multiscale jumps. For and interior interface $e$ of two triangles $K_{i}$ with sampling domains $K_{\delta, i}$ and a boundary interface of a triangle $K$ with sampling domain $K_{\delta}$ we define

$$
\begin{equation*}
\{\bar{\xi}\}=\frac{1}{2}\left(\frac{1}{\left|K_{\delta, 1}\right|} \int_{K_{\delta, 1}} \xi_{1} \mathrm{~d} x+\frac{1}{\left|K_{\delta, 2}\right|} \int_{K_{\delta, 2}} \xi_{2} \mathrm{~d} x\right), \quad\{\bar{\xi}\}=\left(\frac{1}{\left|K_{\delta}\right|} \int_{K_{\delta}} \xi_{1} \mathrm{~d} x\right), \tag{9}
\end{equation*}
$$

respectively, where $\xi$ is an integrable (possibly vector valued) function.
Micro solution. Find for every macro element $K$ the additive contribution to the macro stiffness matrix by computing the micro-problems $u^{h}$ (respectively $v^{h}$ ) on sampling domains $K_{\delta}$ located at quadrature points of the macro element $K$ as follows: find $u^{h}$ such that $\left(u^{h}-u^{H}\right) \in S_{h}$ and

$$
\begin{equation*}
\int_{K_{\delta}} a^{\varepsilon}(x) \nabla u^{h} \cdot \nabla z^{h} \mathrm{~d} x=0 \quad \forall z^{h} \in S_{h}=\left\{z^{h} \in S\left(K_{\delta}\right) ;\left.z^{h}\right|_{T} \in \mathcal{P}^{1}(T), T \in \mathcal{T}_{h}\right\}, \tag{10}
\end{equation*}
$$

where $S\left(K_{\delta}\right)$ determines the coupling condition or boundary conditions used for computing the micro functions $u^{h}, v^{h}$. Several choices are possible; we mention $S\left(K_{\delta}\right)=W_{\text {per }}^{1}\left(K_{\delta}\right)$ referred in what follows as (P), where $W_{\text {per }}^{1}(Y)=$ $\left\{v \in H_{\text {per }}^{1}(Y) ; \int_{Y} v \mathrm{~d} y=0\right\}$ or $S\left(K_{\delta}\right)=H_{0}^{1}\left(K_{\delta}\right)$ referred in what follows as (D).

Variational problem. The macro solution of the DG-HMM is then defined by the following variational problem: find $u_{\mathrm{DG}}^{H} \in V_{H}$ such that

$$
\begin{equation*}
B_{\mathrm{DG}}\left(u_{\mathrm{DG}}^{H}, v^{H}\right)=\int_{\Omega} f v^{H} \mathrm{~d} x, \quad \forall v^{H} \in V_{H}\left(\Omega, \mathcal{T}_{H}\right) . \tag{11}
\end{equation*}
$$

Several remarks are in order. First, the computational saving compared to (7) for a multiscale problem (1) is clear since instead of solving the fine scale on the whole computational domain (as required for (7) with $h<\varepsilon$ ), in the DG-HMM, we only solve the fine scale on sampling domains $K_{\delta}$ usually much smaller than the macro meshsize $H$. Second, we do not require well-defined traces of $a^{\varepsilon}$ on $\partial K$ as was needed in (7). Third, the interface contribution are based on macro functions and averaged micro fluxes already available from the computation of the first term of (8). Fourth, the method is designed for coefficients $a^{\varepsilon}$ of general type. We show below that we do not need special assumptions as for example periodicity for the existence of a solution. However, for the strategy to make sense, scale separation is required in some region of the computational domain where it is applied. For error estimates, we need some explicit expression of the homogenized problem and we therefore treat the periodic case. Error estimates for the FE-HMM in the random case are presented in [11] and are much weaker than for periodic problems. This approach could nevertheless be used to analyse DG-HMM for random coefficients.

## 4. Analysis of the DG-HMM

To state our error estimates, we define $V(H)=V_{H}+H_{0}^{1}(\Omega) \cap H^{2}(\Omega) \subset H^{2}\left(\mathcal{T}_{H}\right)$, with a mesh-dependent norm [6]

$$
\begin{equation*}
\left\|v^{H}\right\|:=\left(\|\nabla v\|_{L^{2}(\Omega)}^{2}+\sum_{K \in \mathcal{T}_{H}} H_{K}^{2}\left|v^{H}\right|_{2, K}^{2}+|v|_{*}^{2}\right)^{1 / 2} \tag{12}
\end{equation*}
$$

where we used the notations $|v|_{m, K}^{2}=\sum_{|\alpha|=m}\left\|\partial^{\alpha} v\right\|_{L^{2}(K)},|v|_{*}^{2}=\sum_{e \in \mathcal{E}}\left\|\mu_{e}^{1 / 2} \llbracket v^{H} \rrbracket\right\|_{L^{2}(e)}^{2}$, and where $\mu_{e}=\alpha H_{e}^{-1}$ as defined above (of course on $V_{H},\left|\left|\left|\left|| |\right.\right.\right.\right.$ reduces to $\left(\|\nabla v\|_{L^{2}(\Omega)}^{2}+|v|_{*}^{2}\right)^{1 / 2}$ ).

Existence of a numerical solution. This can be obtained without more assumption on the microscale. We first recall from Proposition 3.2 in [2] the following energy inequality: let $v^{H} \in V_{H}$ and $l v^{h}$ be the corresponding solution of (P) or (D). Then $\left\|\nabla v^{H}\right\|_{L^{2}\left(K_{\delta}\right)} \leqslant\left\|\nabla v^{h}\right\|_{L^{2}\left(K_{\delta}\right)} \leqslant \frac{\Lambda}{\lambda}\left\|\nabla v^{H}\right\|_{L^{2}\left(K_{\delta}\right)}$, where $\lambda, \Lambda$ are defined in (2). The existence and unicity of $u_{\mathrm{DG}}^{H}$ is then based on the following lemma:

Lemma 4.1. Let $v^{H}, w^{H} \in V_{H}$ and let $v^{h}$ be the solution of $(\mathrm{P})$ or $(\mathrm{D})$ constraint by $v^{H}$. Then,

$$
\begin{equation*}
\sum_{e \in \mathcal{E}} \int_{e}\left\{\overline{a^{\varepsilon} \nabla v^{h}}\right\} \llbracket w^{H} \rrbracket \mathrm{~d} s \leqslant C \alpha^{-1 / 2}\left\|\nabla v^{H}\right\|_{L^{2}(\Omega)}\left|w^{H}\right|_{*} \tag{13}
\end{equation*}
$$

where $\alpha$ is the penalty parameter (see (8)) and where the constant $C$ depends only on the shape regularity of the triangulation $\mathcal{T}_{H}$, the dimension $d$ and the bound in (2).

Proof. Applying the Cauchy-Schwarz inequality to (13) gives

$$
\sum_{\mathcal{E}} \int_{e}\left\{\overline{a^{\varepsilon} \nabla v^{h}}\right\} \llbracket w^{H} \rrbracket \mathrm{~d} s \leqslant \alpha^{-1 / 2}\left(\sum_{\mathcal{E}} H_{e}\left\|\left\{\overline{a^{\varepsilon} \nabla v^{h}}\right\}\right\|_{L^{2}(e)}^{2}\right)^{1 / 2}\left|w^{H}\right|_{*}
$$

A computation using the above energy inequality shows that $\left\|\left\{\overline{a^{\varepsilon} \nabla v^{h}}\right\}\right\|_{L^{2}(e)}^{2} \leqslant C \int_{e}\left|\nabla v_{1}^{H}\right|^{2}+\left|\nabla v_{2}^{H}\right|^{2} \mathrm{~d} s$, where $\nabla v_{i}^{H}$ denotes $\left.\nabla v^{H}\right|_{K_{i}}$. Summing over $\mathcal{E}$, using $\int_{\partial K}\left|\nabla v^{H}\right|^{2} \mathrm{~d} s \leqslant C H_{K}^{-1} \int_{K}\left|\nabla v^{H}\right|^{2} \mathrm{~d} x$ (trace inequality, where $C$ depends on the shape regularity, the dimension $d$ and $H_{K}$ denotes the diameter of the triangle $K$ ), gives the result.

Using this lemma and energy inequality stated above gives us:
Theorem 4.2. There exists a threshold value $\alpha_{\min }$ for the penalty parameter depending only on the shape regularity of the triangulation $\mathcal{T}_{H}$, the dimension $d$ and the bounds in (2), such that for $\alpha \geqslant \alpha_{\min }$ the bilinear form $B_{\mathrm{DG}}$ is uniformly elliptic and bounded on $V_{H} \times V_{H}$ and the problem (11) has a unique solution in $V_{H}$ which satisfies $\left\|u_{\mathrm{DG}}^{H}\right\| \|^{2} \leqslant$ $C\|f\|_{L^{2}(\Omega)}$.

A-priori estimates. Some knowledge of the homogenized problem is needed to derive a-priori estimates and we suppose in what follows that $a^{\varepsilon}=a(x, x / \varepsilon)=a(x, y) Y$-periodic in $y$, where we set $Y=(0,1)^{d}$. In this case the homogenized tensor $a^{0}(x)$ can be computed explicitly with the help of auxiliary functions $\chi^{j}(x, y), j=1, \ldots, d$, obtained for each $x \in \Omega$ as the solution of so-called cell-problems in $Y$ [7]. We assume that $a^{0}$ is piecewise smooth in $\Omega$, that $u_{0}$ is $H^{2}$-regular and that $\chi^{j}(x, y) \in L^{\infty}\left(\Omega, H^{2}(Y)\right), a_{i j}^{\varepsilon}(x, y) \in W^{1, \infty}\left(\Omega, L^{\infty}(Y)\right) \forall i, j=1, \ldots, d$. For such tensor, the optimal boundary conditions for the micro problems are ( P ) and we can choose $K_{\delta}=K_{\varepsilon}$. We will assume that $K_{\varepsilon}$ is centered at $x_{k}$, located for each $K \in \mathcal{T}_{H}$ at the barycenter of $K$. We decompose the error in

$$
\left\|u_{0}-u_{\mathrm{DG}}^{H}\right\| \| \leqslant \underbrace{\left\|u_{0}-\bar{u}_{\mathrm{DG}, 0}^{H}\right\| \|}_{e_{\mathrm{MAC}}}+\underbrace{\left\|\bar{u}_{\mathrm{DG}, 0}^{H}-\bar{u}_{\mathrm{DG}}^{H}\right\| \|}_{e_{\mathrm{MOD}}}+\underbrace{\left\|\bar{u}_{\mathrm{DG}}^{H}-u_{\mathrm{DG}}^{H}\right\| \|}_{e_{\mathrm{MIC}}},
$$

where $\bar{u}_{\mathrm{DG}, 0}^{H}$ is the solution of the standard DG method (6) in $V_{H}$ with a tensor $\left.\bar{a}^{0}(x)\right|_{K}=a^{0}\left(x_{k}\right), \forall K \in \mathcal{T}_{H}$, and $\bar{u}_{\mathrm{DG}}^{H}$ is the solution of the multiscale method (11) with exact micro solution ((10) is computed in $S\left(K_{\varepsilon}\right)$ instead of $S_{h}$ ) and tensor collocated in the slow variables $\bar{a}^{\varepsilon}=\left.\bar{a}(x, x / \varepsilon)\right|_{K}=a\left(x_{k}, x / \varepsilon\right), \forall K \in \mathcal{T}_{H}$, for the micro and macro problems. In the above inequality, $e_{\mathrm{MAC}}, e_{\mathrm{MOD}}, e_{\mathrm{MIC}}$ stand for macro error, modeling error and micro error. Following classical results [6] we have $e_{\mathrm{MAC}}=\left\|\left.\left|u_{0}-u_{\mathrm{DG}, 0}^{H}\| \| \leqslant C H\right| u_{0}\right|_{H^{2}(\Omega)}\right.$.

The modeling error depends on the boundary condition used in the micro problem. Using $\omega_{K_{\varepsilon}} \int_{K_{\varepsilon}} \bar{a}^{\varepsilon} \nabla u \cdot \nabla v \mathrm{~d} x=$ $\int_{K} \bar{a}^{0} \nabla \bar{u}_{\mathrm{DG}}^{H} \cdot \nabla v^{H} \mathrm{~d} x$ [4, Appendix] and $\frac{1}{\left|K_{\varepsilon}\right|} \int_{K_{\varepsilon}}\left(\bar{a}^{\varepsilon} \nabla v\right) \mathrm{d} x=\bar{a}^{0}\left(x_{k}\right) \nabla v^{H}$ [3, Section 2.3] it can be shown that:

Lemma 4.3. For the boundary condition $(\mathrm{P}), e_{\mathrm{MOD}} \equiv 0$.

Using a Strang like lemma, $e_{\text {MIC }}$ can be estimated as follows:
Lemma 4.4. $\left\|\bar{u}_{\mathrm{DG}}^{H}-u_{\mathrm{DG}}^{H}\right\| \| \leqslant\left(C_{p}\left(\frac{h}{\varepsilon}\right)^{2}+C_{f}\left(\frac{h}{\varepsilon}\right)+\widetilde{C} \varepsilon\right)$, where the constants $C_{p}, C_{f}, \widetilde{C}$ are independent of $\varepsilon$.
The first term in the above error comes from the error in the first member in (8) and its estimate follows [2, Lemma 3.3], the second term comes from the error in the flux in (8) while the last term is a contribution to the error from the discrepancy between $a(x, x / \varepsilon)$ and $\bar{a}(x, x / \varepsilon)$. Combining the above lemmas and using the discrete Poincaré-Friedrich inequality $\|v\|_{L^{2}(\Omega)} \leqslant C\left(\|v\|_{L^{2}(\Omega)}^{2}+|v|_{*}^{2}\right)$ [5, Lemma 2.1] gives the following theorem:

## Theorem 4.5.

$$
\begin{equation*}
\left\|u_{0}-u_{\mathrm{DG}, 0}^{H}\right\|\left\|\leqslant C\left(H+\frac{h}{\varepsilon}+\varepsilon\right), \quad\right\| u_{0}-u_{\mathrm{DG}, 0}^{H} \|_{L^{2}(\Omega)} \leqslant C\left(H^{2}+\frac{h}{\varepsilon}+\varepsilon\right) . \tag{14}
\end{equation*}
$$

If the numerical method (8) is used with a collocated tensor $\bar{a}^{\varepsilon}$ instead of $a^{\varepsilon}$ (for the micro and macro methods) then the third term in the estimations (14) vanishes and we thus have a robust, i.e., independent of $\varepsilon$, convergence rate (notice that $\frac{h}{\varepsilon} \simeq N_{\text {mic }}^{-1 / d}$ does only depend on $N_{\text {mic }}$, the micro degrees of freedom and not on $\varepsilon$ ).

Detailed proofs together with procedure to reconstruct microscopic information from the known solution on sampling domains as well as numerical examples will be presented in a forthcoming publication.

## Acknowledgement

Assyr Abdulle is supported by an EPSRC Advanced Fellowship EP/E05207X/1.

## References

[1] J. Aarnes, B.-O. Heimsund, Multiscale discontinuous Galerkin methods for elliptic problems with multiple scales, in: Lecture Notes Comput. Sci. Eng., vol. 44, Springer-Verlag, 2006, pp. 1-20.
[2] A. Abdulle, On a-priori error analysis of fully discrete heterogeneous multiscale FEM, SIAM Multiscale Model. Simul. 4 (2) (2005) 447-459.
[3] A. Abdulle, W. E, Finite difference HMM for homogenization problems, J. Comput. Phys. 191 (1) (2003) 18-39.
[4] A. Abdulle, C. Schwab, Heterogeneous multiscale FEM for diffusion problem on rough surfaces, SIAM Multiscale Model. Simul. 3 (1) (2005) 195-220.
[5] D. Arnold, An interior penalty finite element method with discontinuous element, SIAM J. Numer. Anal. 19 (1982) 742-760.
[6] D. Arnold, F. Brezzi, B. Cockburn, D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, SIAM J. Numer. Anal. 5 (2002) 1749-1779.
[7] A. Bensoussan, J.-L. Lions, G. Papanicolaou, Asymptotic Analysis for Periodic Structures, North-Holland, Amsterdam, 1978.
[8] S. Chen, W. E, C. Shu, The heterogeneous multiscale method based on the discontinuous Galerkin method for hyperbolic and parabolic problems, SIAM Multiscale Model. Simul. 3 (4) (2005) 871-894.
[9] E. De Giorgi, S. Spagnolo, Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine, Boll. Un. Mat. Ital. 4 (8) (1973) 391-411.
[10] W. E, B. Engquist, The heterogeneous multi-scale methods, Commun. Math. Sci. 1 (1) (2003) 87-132.
[11] W. E, P. Ming, P. Zhang, Analysis of the heterogeneous multiscale method for elliptic homogenization problems, J. Amer. Math. Soc. 18 (1) (2004) 121-156.


[^0]:    E-mail address: a.abdulle@ed.ac.uk.

    1631-073X/\$ - see front matter © 2007 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
    doi:10.1016/j.crma.2007.11.029

