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Complex Analysis

Algebraic analysis of Hermitian monogenic functions

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Abstract

In this Note we present an algebraic analysis of the system of differential equations described by the Hermitian Dirac operators, which are two linear first order operators invariant with respect to the action of the unitary group, both in the case of one and several variables. ***To cite this article: A. Damiano et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

L'analyse algébrique des fonctions monogènes Hermitiennes. Nous présentons l'analyse algébrique du système associé aux opérateurs de Dirac Hermitiens. Ceux-ci sont deux opérateurs linéaires du premier ordre, invariant sous l'action du groupe unitaire. Nous étudions le cas d'une et des plusieurs variables. ***Pour citer cet article : A. Damiano et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

Recently, there has been an increasing interest in the notion of monogenicity in the Hermitian setting. Monogenic functions are defined as Clifford algebra valued (or spinor valued) functions belonging to the kernel of the Dirac operator, which is a first order elliptic differential operator invariant with respect to the induced action of the orthogonal Lie algebra $\mathfrak{so}(m)$ on spinor fields. This setting can be extended to the case of arbitrary Riemannian spin manifolds on which the Dirac operator arises as a conformally invariant operator acting on sections of a spinor bundle. By introducing a complex structure, the underlying symmetry can be reduced from $\mathfrak{so}(2n)$ to $\mathfrak{u}(n)$ (or its complexified semisimple part $\mathfrak{sl}(n)$). This leads to two new differential operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}}^{\dagger}$, called Hermitian Dirac operators, defined on suitable bundles over complex Kähler manifolds, which are locally homeomorphic to $\mathbb{C}^n$. The kernel of these operators defines the Hermitian monogenic functions. In view of the fact that the $\mathfrak{so}(2n)$-irreducible spinor spaces decompose into the so-called homogeneous spinor spaces carrying the fundamental representations for $\mathfrak{sl}(n)$, it is important to note that by restricting the values of the functions to these homogeneous spinor spaces, the system defining Hermitian

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monogenic functions splits into several systems of equations in several complex variables, among which the system defining holomorphic or anti-holomorphic functions. In this Note we announce the algebraic analysis of the system associated to Hermitian monogenic functions and we study the resolution in the one variable case as well as some consequences. We also further generalize the theory to the setting of quaternionic Hermitian monogenic functions. The proofs of the results in this Note will be presented elsewhere (see [4,5]).

2. Hermitian monogenic functions: the complex case

Let $\mathbb{R}^{2n} = \mathrm{span}_{\mathbb{R}}(e_1, \ldots, e_{2n})$ be the real orthogonal space endowed with the symmetric real-bilinear form $\mathcal{B}_{\mathbb{R}}(e_i, e_j) = -\delta_{ij}$, its complexification $\mathbb{C}^{2n}$ and its associated Clifford algebra $\mathbb{C}_{2n}$, where $e_i e_j + e_j e_i = -2\delta_{ij}$. We can select a so-called *complex structure* J on $\mathbb{R}^{2n}$, by choosing a specific automorphism $J \in \mathrm{SO}(2n)$ such that $J^2 = -\mathbf{1}_{\mathbb{R}^{2n}}$: we define $J[e_j] = -e_{j+n}$, $J[e_{j+n}] = e_j$, for $j = 1, \ldots, n$. Using the relation $\mathbf{1}_{2n} = \frac{1}{2}(\mathbf{1}_{2n} + iJ) + \frac{1}{2}(\mathbf{1}_{2n} - iJ) = \pi^+ + \pi^-$, the complex vector space $\mathbb{C}^{2n}$ can be written as the direct sum of two maximally isotropic subspaces $W^{\pm} = \pi^{\pm}(\mathbb{C}^{2n})$.

Definition 2.1. The Witt basis for $\mathbb{C}^{2n}$ is defined by $\mathfrak{f}_j = +\pi^+[e_j] = \frac{1}{2}(e_j - ie_{j+n})$ and $\mathfrak{f}_j^\dagger = -\pi^-[e_j] = \frac{1}{2}(e_j + ie_{j+n})$, for $1 \leqslant j \leqslant n$.

Remark 1. In terms of the Witt basis, the multiplication rules for $\mathbb{C}_{2n}$ become $\mathfrak{f}_j\mathfrak{f}_i + \mathfrak{f}_i\mathfrak{f}_j = \mathfrak{f}_j^\dagger\mathfrak{f}_i^\dagger + \mathfrak{f}_i^\dagger\mathfrak{f}_j^\dagger = 0$ and $\mathfrak{f}_j\mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger\mathfrak{f}_j = \delta_{jk}$.

Definition 2.2. Given the vector variable $\underline{X} = \sum_{j=1}^{2n} e_j X_j \in \mathbb{R}^{2n}$ and the Dirac operator $\partial_{\underline{X}} = \sum_{j=1}^{2n} e_j \partial_{X_j}$, we define two complex vector variables $\underline{z} = +\pi^+[\underline{X}] = \sum_{j=1}^{n} \mathfrak{f}_j z_j$, $\underline{z}^\dagger = -\pi^-[\underline{X}] = \sum_{j=1}^{n} \mathfrak{f}_j^\dagger z_j^c$, with $z_j = X_j + iX_{j+n}$, $z_j^c = X_j - iX_{j+n}$, $j = 1, \ldots, n$, and two complex Hermitian Dirac operators

$$2\partial_{\underline{z}}^\dagger = +\pi^+[\partial_{\underline{X}}] = 2\sum_{j=1}^{n} \mathfrak{f}_j \partial_{z_j^c}, \qquad 2\partial_{\underline{z}} = -\pi^-[\partial_{\underline{X}}] = 2\sum_{j=1}^{n} \mathfrak{f}_j^\dagger \partial_{z_j},$$

where $\partial_{z_j^c}$ and ∂_{z_j} denote the Cauchy–Riemann operator and its conjugate, in the variables z_j.

Definition 2.3. Let $f(\underline{z}, \underline{z}^\dagger)$ be a smooth function on $\mathbb{R}^{2n}$ with values in the complex Clifford algebra $\mathbb{C}_{2n}$. Then f is called Hermitian monogenic (in short, h-monogenic) if it satisfies the system $\partial_{\underline{z}} f = \partial_{\underline{z}}^\dagger f = 0$. Let $\underline{z}_i = \sum_j \mathfrak{f}_j z_{ij}$ and $\underline{z}_i^\dagger = \sum_j \mathfrak{f}_j^\dagger z_{ij}^c$, $i = 1, \ldots, k$, and let $\partial_{\underline{z}_i}^\dagger$, $\partial_{\underline{z}_i}$ be the corresponding operators. A function $f(\underline{z}_1, \underline{z}_1^\dagger, \ldots, \underline{z}_k, \underline{z}_k^\dagger)$ with values in $\mathbb{C}_{2n}$ is said to be h-monogenic if it is in the kernel of $\partial_{\underline{z}_i}^\dagger$ and $\partial_{\underline{z}_i}$ for all $i = 1, \ldots, k$.

For the basic results on h-monogenic functions we refer the reader to [1,2,8]. Defining $I_j = \mathfrak{f}_j\mathfrak{f}_j^\dagger$ and $I = I_1 \cdots I_n$ we have that $\mathbb{C}_{2n}I$ is a model for the standard complex spinor space $\mathbb{S}$. If we denote by $\Lambda_j^\dagger$ the span of j-vectors $\mathfrak{f}_k^\dagger \in W^-$ and by $\mathbb{S}_j = \Lambda_j^\dagger I$, then we have the decomposition $\mathbb{S} = \bigoplus_{j=0}^{n} \mathbb{S}_j$. With respect to $\mathbb{S} = \bigoplus_{j=0}^{n} \mathbb{S}_j$ we can represent the Dirac operators as the following block matrices with entries in $\mathbb{C}[\underline{z}, \underline{z}^\dagger]$ (with an abuse we will indicate the matrices with the same symbol as the operators)

$$\partial_{\underline{z}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ d_1 & 0 & \cdots & 0 & 0 \\ 0 & d_2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n & 0 \end{bmatrix} \quad \text{and} \quad \partial_{\underline{z}}^\dagger = \begin{bmatrix} 0 & \delta_0 & 0 & \cdots & 0 \\ 0 & 0 & \delta_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \delta_{n-1} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{1}$$

where the i, j-th block in $\partial_{\underline{z}}$ is either a $\binom{n}{i}$ times $\binom{n}{j}$ matrix of zeroes or the restriction of the Dirac operator $d_i := \pi_{\mathbb{S}_i} \circ [\partial_{\underline{z}}]_{|\mathbb{S}_{i-1}} : \mathbb{S}_{i-1} \to \mathbb{S}_i$, and similarly for $\partial_{\underline{z}}^\dagger$ with $\delta_i := \pi_{\mathbb{S}_i} \circ [\partial_{\underline{z}}^\dagger]_{|\mathbb{S}_{i+1}} : \mathbb{S}_{i+1} \to \mathbb{S}_i$. In the sequel, we will consider the symbols of the operators $\partial_{\underline{z}}$, $\partial_{\underline{z}}^\dagger$. Their entries are in the ring of polynomials $R = \mathbb{C}[y_1, \dots, y_m]$ for a suitable set of variables $y_1, \dots, y_m$. Let us denote by P the Fourier transform of the matrix $P(D)$ made by the two blocks $\partial_{\underline{z}}$, $\partial_{\underline{z}}^\dagger$ in a column. Let $\mathcal{M}_P$ be the cokernel of P. We say that $\mathcal{M}_P$ is the module associated to the Hermitian Dirac system. The description above allows to prove, using Gröbner basis techniques, the following result:

Theorem 2.4. *The free resolution of the module $\mathcal{M}_P$ associated to the Hermitian Dirac system*

$$\begin{cases} \partial_{\underline{z}} f = a, \\ \partial_{\underline{z}}^\dagger f = b \end{cases} \tag{2}$$

in dimension $m = 2n$ is linear of length n. All the maps in the resolutions can be explicitly described. In particular: the first syzygies are described by the relations

$$\begin{cases} d_i a|_{\mathbb{S}_{i-1}} = 0, & 2 \leqslant i \leqslant n, \\ \delta_i b|_{\mathbb{S}_{i+1}} = 0, & 0 \leqslant i \leqslant n-2, \end{cases} \tag{3}$$

which form $2^n - n - 1$ complex relations among the scalar components of a and b. The Betti numbers in the resolution are given by $\beta_0 = 2^n$ and $\beta_i = 2\left(\sum_{j=i}^n \binom{n}{j}\right)$, $1 \leqslant i \leqslant n$.

Remark 2. It is possible to completely describe the first syzygies in the case of the Hermitian system in several variables (see [4]). It is interesting to note that the radial relations (see [8]) are never enough to describe the syzygies. We recall that in the orthogonal case the radial relations are enough in the case in which the number k of Dirac operators and the dimension m of the Clifford algebra satisfy the constraint $m \geqslant 2k - 1$ (see [3]).

Remark 3. Relations (3) imply some restrictions on the scalar values of a and b (in general, for the system in k variables, we have constraints on the data a_i and b_i), namely $a|_{\mathbb{S}_0} = b|_{\mathbb{S}_n} = 0$ (resp. $a_i|_{\mathbb{S}_0} = b_i|_{\mathbb{S}_n} = 0$ for all i).

Remark 4. In the orthogonal case, when dealing with only one vector variable $\underline{X} \in \mathbb{R}^{2n}$, the matrix associated to the operator is square and the complex ends after one step while in the Hermitian case the system corresponds to a matrix which is not square and the complex has length depending on the dimension of the algebra in which it is embedded. When the complex associated to a differential operator has length greater than 1, like the case of systems of several orthogonal Dirac operators, Hartogs' type phenomena occur. However, this is not the case for the Hermitian system in one variable, as the following result shows (see [4]):

Theorem 2.5. *Let K be a compact convex subset of an open set $U \subseteq \mathbb{R}^{2n}$, and let f be a Hermitian monogenic function on $U \setminus K$. Then f cannot, in general, be extended to a Hermitian monogenic function on U.*

Remark 5. In the several variable case, the Hartogs' phenomenon occurs (see [4]).

A remarkable consequence of the constraints on the data a_i, b_i is the following: while in the orthogonal case the study of the Dirac equation $\partial_{\underline{X}} f = g$ is equivalent to the analysis of one of the two reductions into even and odd parts $\partial_{\underline{X}}[f]^+ = [g]^-$, $\partial_{\underline{X}}[f]^- = [g]^+$, according to the splitting $\mathbb{C}_m = \mathbb{C}_m^{\text{even}} \oplus \mathbb{C}_m^{\text{odd}}$ which allows a reduction of the complexity of the problem, in this case we have:

Theorem 2.6. *The Hermitian system is equivalent to the reduction to its even or odd part only when* $m \equiv 2 \bmod(4)$.

3. Hermitian monogenic functions: the quaternionic case

Let us now consider the quaternionic version of the Hermitian system. To this end, we will consider $\mathbb{H}_{4m} := \mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}_{4m}$ endowed with the so called Witt basis now defined:

Definition 3.1. The quaternionic Witt basis of $\mathbb{H}_{4m}$, is given by $\{f_\ell, f_\ell^\alpha, f_\ell^\beta, f_\ell^\gamma\}$, $\ell = 1, \dots, m$, where

$$f_\ell = e_{1+4(\ell-1)} - ie_{2+4(\ell-1)} - je_{3+4(\ell-1)} - ke_{4\ell}, \qquad f_\ell^\alpha = e_{1+4(\ell-1)} + ie_{2+4(\ell-1)} - je_{3+4(\ell-1)} - ke_{4\ell},$$
$$f_\ell^\beta = e_{1+4(\ell-1)} - ie_{2+4(\ell-1)} + je_{3+4(\ell-1)} - ke_{4\ell}, \qquad f_\ell^\gamma = e_{1+4(\ell-1)} - ie_{2+4(\ell-1)} - je_{3+4(\ell-1)} + ke_{4\ell}.$$

We can give the notion of monogenic Hermitian functions by introducing the operators $\partial_{\underline{q}} = \sum_{\ell=1}^n f_\ell \partial_{q_\ell}$ where $\partial_{q_\ell} = \partial_{x_{1+4(\ell-1)}} + i\partial_{x_{2+4(\ell-1)}} + j\partial_{x_{3+4(\ell-1)}} + k\partial_{x_{4\ell}}$, $\ell = 1, \dots, m$, is a Cauchy–Fueter type operator and its variations $\partial_{\underline{q}}^\alpha = \sum_{\ell=1}^n f_\ell^\alpha \partial_{q_\ell}^\alpha$ and similarly for $\partial_{\underline{q}}^\beta$, $\partial_{\underline{q}}^\gamma$, where α, β, γ are suitable involutions acting on q (see [7]).

Definition 3.2. Let U be an open set in $\mathbb{R}^{4m}$. A function $f : U \subseteq \mathbb{R}^{4m} \to \mathbb{H}_{4m}$ is called quaternionic Hermitian (in short q-Hermitian) monogenic it satisfies all the four equations

$$\partial_{\underline{q}} f = \partial_{\underline{q}}{}^\alpha f = \partial_{\underline{q}}{}^\beta f = \partial_{\underline{q}}{}^\gamma f = 0.$$

There are different, but equivalent, definitions of q-Hermitian functions (see [6,5,7]). In particular, see [6], it is possible to adopt an equivalent definition in which the four operators involved are $\mathrm{sp}_{2n}(\mathbb{C})$-invariant.

Remark 6. The resolution associated to the q-Hermitian system is of length n (see [5]); in the low dimensional cases it is linear (see [7]) and in [5] we are using the methods used in the complex case, to show that it is linear for any value of the dimension. If we denote by $\mathcal{M}_P$ the module associated to the q-Hermitian system, we have the following:

Theorem 3.3. $\mathrm{Ext}^1(\mathcal{M}_P, R) \neq 0$.

Corollary 3.4. *Let K be a compact convex subset of an open set $U \subseteq \mathbb{R}^{4n}$, and let f be a q-Hermitian monogenic function on $U \setminus K$. Then f cannot, in general, be extended to a q-Hermitian monogenic function on U.*

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