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Differential Geometry

Approximating $W^{2,2}$ isometric immersions

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Abstract

Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain and set $W^{2,2}_{\text{iso}}(S;\mathbb{R}^3) = \{u \in W^{2,2}(S;\mathbb{R}^3)\colon\ (\nabla u)^T(\nabla u) = \text{Id a.e.}\}$. Under an additional regularity condition on the boundary ∂S (which is satisfied if it is piecewise continuously differentiable) we prove that the $W^{2,2}$ closure of $W^{2,2}_{\text{iso}}(S;\mathbb{R}^3) \cap C^\infty(\bar{S};\mathbb{R}^3)$ agrees with $W^{2,2}_{\text{iso}}(S;\mathbb{R}^3)$. ***To cite this article: P. Hornung, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Approximation de $W^{2,2}$ par des immersions isométriques. Soient $S \subset \mathbb{R}^2$ un domaine lipschitzien borné et $W^{2,2}_{\text{iso}}(S;\mathbb{R}^3)$ l'ensemble $W^{2,2}_{\text{iso}}(S;\mathbb{R}^3) = \{u \in W^{2,2}(S;\mathbb{R}^3)\colon\ (\nabla u)^T(\nabla u) = \text{Id p.p.}\}$. Sous une hypothèse supplémentaire de régularité sur la frontière ∂S (qui est satisfaite dans le cas où ∂S est continument différentiable par morceaux), nous prouvons que l'adhérence $W^{2,2}$ de $W^{2,2}_{\text{iso}}(S;\mathbb{R}^3) \cap C^\infty(\bar{S};\mathbb{R}^3)$ est $W^{2,2}_{\text{iso}}(S;\mathbb{R}^3)$. ***Pour citer cet article : P. Hornung, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

In this Note we study the question whether isometric immersions u from a two-dimensional domain S into $\mathbb{R}^3$ that belong to the Sobolev class $W^{2,2}$ can be approximated by isometric immersions which are smooth up to the boundary of S. This is an approximation result for Sobolev mappings which respects nonconvex constraints on the derivatives, see e.g. [1] for a broader perspective onto such problems. Besides this intrinsic interest, our main motivation arises from nonlinear plate theory, where $W^{2,2}$ isometric immersions form the natural class of admissible functions. For a bounded domain $S \subset \mathbb{R}^2$ we define $W^{2,2}_{\text{iso}}(S;\mathbb{R}^3) = \{u \in W^{2,2}(S;\mathbb{R}^3)\colon\ (\nabla u)^T(\nabla u) = \text{Id a.e.}\}$. This class agrees with the set of finite energy deformations in Kirchhoff's plate theory as derived in [3,4]. The density result presented here is a key ingredient in the rigorous derivation of related thin-film theories from nonlinear three dimensional elasticity, compare e.g. the recent work [2]. In the derivation of Kirchhoff's plate theory given in [9,10], the author raised the question whether one can approximate $W^{2,2}$ isometric immersions by smooth ones or not, and he outlined a proof for the upper bound (alternative to that in [4]) provided such an approximation result is true.

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The problem studied here was previously addressed in [8], where it is shown that if $S \subset \mathbb{R}^2$ is convex and has a smooth boundary then there are approximating isometric immersions which are of class C^2 in the interior. Both in that work and here the developability theorem from [6] plays a crucial role. It states that, away from planar points, the image of a $W^{2,2}$ isometric immersion is a developable surface. (Strictly speaking, his result applies to $W^{2,\infty}_{\text{iso}}$, but essentially the same proof also works in the $W^{2,2}_{\text{iso}}$ case, see [8].) Thus the level sets of ∇u provide a natural fibration (called ruling in the sequel) of the domain into straight line segments $(\beta^-, \beta^+) \subset S$ with endpoints β^-, β^+ in ∂S.

The smoothing process will change the geometry of these lines. But to obtain an approximating isometric immersion which is well defined on the whole domain S the modified lines should also not intersect within S. If the perturbation is small, then the modified lines will only intersect near the boundary ∂S. If S is convex (or strictly star-shaped) then its closure is contained in dilated versions of S, so dilating the domain together with the modified ruling results in an isometric immersion that is well defined on S. Unfortunately, generic domains lack this dilation property, so here a totally different approach is necessary: One has to subdivide the domain into pieces compatible with the ruling and construct local approximations on each piece. Each of them must satisfy *prescribed boundary conditions* on the parts of the boundary which have been created by the subdivision process. Since, due to developability, isometric immersions are rather rigid, it is a priori not clear whether the class of $W^{2,2}$ isometric immersions satisfying these boundary conditions will contain a smooth isometric immersion at all. However, we show that it even contains smooth isometric immersions that can be extended as an isometric immersion to a larger domain. Notice that even after restricting to simply connected subdomains, nonconvexity still causes further effects which are not seen in the convex case. They are related to the fact that the rulings may intersect the boundary tangentially. A global difficulty caused by holes in the domain is that the set where u is locally affine is geometrically much less constrained than in the convex case and must therefore be handled by more sophisticated topological arguments. In this Note we introduce some important notions and give a survey over the main ideas. Detailed proofs will be published elsewhere [5].

A bounded Lipschitz domain $S \subset \mathbb{R}^2$ is said to satisfy condition $(*)$ if there is a closed subset $\Sigma \subset \partial S$ with $\mathcal{H}^1(\Sigma) = 0$ such that the outer unit normal $\hat{\nu}$ exists and is continuous on $\partial S \setminus \Sigma$. Our main result reads as follows:

Theorem 1. *Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain satisfying condition $(*)$. Then the $W^{2,2}$ closure of $W^{2,2}_{\text{iso}}(S;\mathbb{R}^3) \cap C^\infty(\bar{S};\mathbb{R}^3)$ agrees with $W^{2,2}_{\text{iso}}(S;\mathbb{R}^3)$.*

2. Quasi-developable mappings

We denote by $S \subset \mathbb{R}^2$ a bounded Lipschitz domain. For $\mu \in S^1$ and $x \in S$ we denote by $[x]_\mu$ the connected component of $(x + \operatorname{span} \mu) \cap S$ that contains x. A mapping f from a subset $X \subset S$ into $\mathbb{R}^P$ is called *developable* if for every point $x \in X$ there exists $q_f(x) \in S^1$, unique up to a sign, such that f is constant on $[x]_{q_f(x)}$, and $[x]_{q_f(x)} \cap [y]_{q_f(y)} \neq \emptyset$ implies $[x]_{q_f(x)} = [y]_{q_f(y)}$. The mapping q_f is called a *ruling* for f. If X is a domain with $\bar{X} \subset S$ then q_f can be chosen to be Lipschitz on X. In the sequel we write $[x]$ instead of $[x]_{q_f(x)}$. For $f \in C^0(S;\mathbb{R}^P)$, $P \in \mathbb{N}$, $P \geqslant 2$, we introduce $C_f = \{x \in S\colon\ f \text{ is constant in a neighbourhood of } x\}$. A mapping $f \in C^0(S;\mathbb{R}^P)$ is called *quasi-developable* if $f|_{S\setminus C_f}$ is developable. From now on f denotes such a mapping. By means of a nontrivial analysis [5] of the set C_f one can show that for the purpose of proving Theorem 1 one may assume without loss of generality that C_f and $S \setminus C_f$ consist of finitely many connected components.

For $x \in S$ and $\mu \in S^1$ we define $\nu(x,\mu) = \inf\{\theta > 0\colon\ x + \theta\mu \notin S\}$. Let $\Gamma \in W^{2,\infty}([0,T];S)$ be arclength-parametrized, set $N = (\Gamma')^\perp$, denote by $\kappa = \Gamma'' \cdot N$ the curvature of Γ and define the Frénet frame $R = (\Gamma', N)^T$. The Frénet equations read $R' = \kappa(e_1 \otimes e_2 - e_2 \otimes e_1)R$. For $t \in [0,T]$ and $* \in \{+,-\}$ set $s^*_\Gamma(t) = *\nu(\Gamma(t), *N(t))$ and $\beta^*_\Gamma(t) = \Gamma(t) + s^*_\Gamma(t)N(t)$. Abusing notation, set $[\Gamma(t)] = [\Gamma(t)]_{N(t)}$. For $J \subset [0,T]$ we introduce the notation $[\Gamma(J)] = \bigcup_{t\in J}[\Gamma(t)]$. We will assume without loss of generality that $T^{-1} > \|\kappa\|_{L^\infty(0,T)}$, so Γ is injective. The curve Γ is called *simple* if there are two Lipschitz functions α^+, α^- in local coordinate systems such that $\beta^*_\Gamma([0,T]) \subset \operatorname{graph}\alpha^* \subset \partial S$ for $* = +,-$. For simple curves we introduce the sets $D^*_\Gamma = \{t \in [0,T]\colon\ (\alpha^*)'((\beta_\Gamma)_1(t)) \text{ exists and } N_2(t) = (\alpha^*)'((\beta_\Gamma)_1(t))N_1(t)\}$ where the closure of $[\Gamma(t)]$ is tangential to ∂S (the subindices refer to the local coordinates of $\operatorname{graph}\alpha^*$). We say that Γ is *transversal* on a set $J \subset [0,T]$ if $(D^+_\Gamma \cup D^-_\Gamma) \cap J = \emptyset$. The curve Γ is called *admissible* if the line segments $(\beta^-_\Gamma(t), \beta^+_\Gamma(t))$ are disjoint for different t. One can show [5] that if S is simply connected and Γ is simple, then Γ is admissible if and only if $1 - s^*(t)\kappa(t) \geqslant 0$ for $* = +,-$ and a.e. $t \in (0,T)$, thus providing a local criterion for admissibility. We say that Γ is *strictly admissible* if there exists $\delta > 0$ such that $1 - s^*(t)\kappa(t) \geqslant \delta$ for $* = +, -$ and a.e. t. If f is developable in a neighbourhood of $\Gamma([0,T]) \subset S$ and if Γ solves the ODE $\Gamma'(t) = -q_f^{\perp}(\Gamma(t))$ then Γ is called an *f-integral curve*.

3. Modifying $W^{2,2}$ isometric immersions

The above discussion is linked to Theorem 1 by the developability theorem presented in [6,8,7], which says that if $S \subset \mathbb{R}^2$ is a bounded Lipschitz domain and $u \in W^{2,2}_{\text{iso}}(S;\mathbb{R}^3)$ then $\nabla u \in C^0(S;\mathbb{R}^{3\times 2})$ is quasi-developable. (Continuity in the interior is shown in [7], where it is attributed to Kirchheim.)

From now on we fix a bounded Lipschitz domain $S \subset \mathbb{R}^2$ that satisfies condition $(*)$ and we fix $u \in W^{2,2}_{\text{iso}}(S;\mathbb{R}^3)$ and a simple ∇u-integral curve $\Gamma \in W^{2,\infty}([0,T];S)$. Following [8] we introduce the space curve $\gamma = u \circ \Gamma$ and we set $v = (\nabla u \circ \Gamma)N$ and $n = \gamma' \wedge v$. One can prove [5] that $r := (\gamma', v, n)^T$ is in $W^{1,2}((0,T);\mathrm{SO}(3))$. It satisfies

$$r' = \big(\kappa(e_1 \otimes e_2 - e_2 \otimes e_1) + \kappa_n(e_1 \otimes e_3 - e_3 \otimes e_1)\big)r, \tag{1}$$

where $\kappa_n := \gamma'' \cdot n \in L^2(0,T)$, and $\int_{[\Gamma(0,T)]} |\nabla^2 u(x)|^2\,\mathrm{d}x = \int_0^T \kappa_n^2(t)\big(\int_{s_\Gamma^-(t)}^{s_\Gamma^+(t)} \frac{1}{1-s\kappa(t)}\,\mathrm{d}s\big)\,\mathrm{d}t$. The integrand in parentheses is bounded from below by a positive constant, so if $u \in W^{2,2}$ then $\kappa_n \in L^2$. Conversely, given $\bar\Gamma \in W^{2,\infty}([0,\bar T];S)$ and $\bar\kappa_n \in L^2(0,\bar T)$ one can *define* a mapping $(\bar\Gamma, \bar\kappa_n)$ by setting $(\bar\Gamma,\bar\kappa_n)(\bar\Gamma(t) + sN(t)) = \bar\gamma(t) + sv(t)$ for $t \in (0,T)$, $s \in \mathbb{R}$. If $\bar\Gamma$ is strictly admissible then $(\bar\Gamma,\bar\kappa_n)$ is well defined and lies in $W^{2,2}_{\text{iso}}([\bar\Gamma(0,\bar T)];\mathbb{R}^3)$, see [5]. With this notation we have $u|_{[\Gamma(0,T)]} = (\Gamma,\kappa_n)$. We write $(\Gamma,\kappa_n) \sim (\bar\Gamma,\bar\kappa_n)$ provided $[\Gamma(0,T)] = [\bar\Gamma(0,\bar T)]$ and (Γ,κ_n) and $(\bar\Gamma,\bar\kappa_n)$ agree up to the first derivatives on $S \cap \partial[\Gamma(0,T)]$. One can prove [5]: If Γ and $\bar\Gamma$ are transversal, if $\bar\gamma(0) = \gamma(0)$, $\bar r(0) = r(0)$, $\bar\Gamma(0) = \Gamma(0)$, $\bar R(0) = R(0)$ and if

$$\bar\gamma(\bar T) - \gamma(T) = \big(v(T) \otimes N(T)\big)\big(\bar\Gamma(\bar T) - \Gamma(t)\big), \qquad \bar r(\bar T) = r(T), \tag{2}$$

$$\bar\Gamma(\bar T) - \Gamma(T) \parallel N(T), \qquad \bar R(\bar T) = R(T) \tag{3}$$

are satisfied then $(\Gamma,\kappa_n) \sim (\bar\Gamma,\bar\kappa_n)$. (The frame $\bar r$ is defined by solving (1) with $\bar\kappa = \bar\Gamma'' \cdot \bar N$ and $\bar\kappa_n$ instead of κ and κ_n, respectively. We also set $\bar\gamma' := \bar r^T e_1$.) The transversality assumption is used to ensure that $S \cap \partial[\Gamma(0,T)] = [\Gamma(0)] \cup [\Gamma(T)]$; compare Fig. 1 (right).

Lemma 2 *(Main lemma). Let $(\Gamma,\kappa_n) \in W^{2,2}_{\text{iso}}([\Gamma(0,T)];\mathbb{R}^3)$ and let $\delta > 0$. Then there exist $\delta' > 0$ and $(\bar\Gamma,\bar\kappa_n)$ such that $(\bar\Gamma,\bar\kappa_n) \sim (\Gamma,\kappa_n)$, $(\bar\Gamma,\bar\kappa_n) \in C^\infty_{\text{iso}}(\overline{[\Gamma(0,T)]};\mathbb{R}^3)$, $(\bar\Gamma,\bar\kappa_n)$ is affine on $B_{\delta'}(S \cap \partial[\Gamma(0,T)])$ and $\|(\bar\Gamma,\bar\kappa_n) - (\Gamma,\kappa_n)\|_{W^{2,2}([\Gamma(0,T)];\mathbb{R}^3)} < \delta$.*

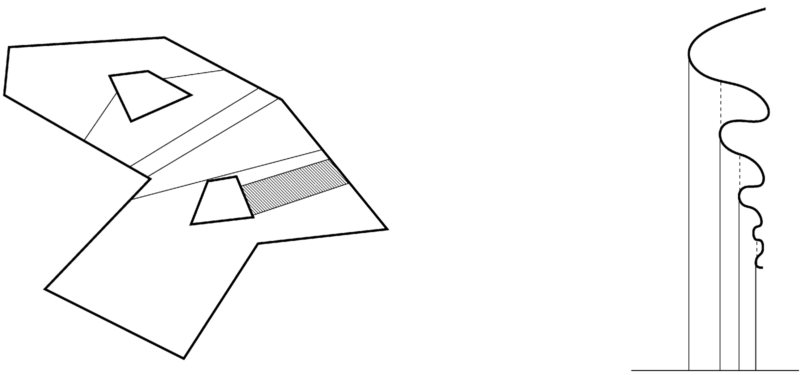

Fig. 1. Left: The domain S with some rulings. It is subdivided into simply connected subdomains like the dashed one. Right: Zoom-in of the dashed region from the left. The bottom line is Γ, the bold wavy line is part of the boundary ∂S. Some rulings intersect tangentially, whence $S \cap \partial[\Gamma(0,T)] \neq [\Gamma(0)] \cup [\Gamma(T)]$.

Fig. 1. Gauche : Le domaine S avec quelques lignes de niveau de ∇u. Il est subdivisé en sous-domaines simplement connexes comme celui hachuré. Droite : Zoom dans la région hachurée. La ligne du bas est Γ, la ligne ondulée est une partie de la frontière ∂S. Quelques lignes de niveau intersectent ∂S tangentiellement, de sorte que $S \cap \partial[\Gamma(0,T)] \neq [\Gamma(0)] \cup [\Gamma(T)]$.

The proof consists of three key steps. A nontrivial argument [5] allows one to assume without loss of generality that Γ is transversal on $[0, T]$. *Step 1.* Using condition $(*)$, by a nontrivial procedure one modifies (Γ, κ_n) such that the resulting mapping $(\bar{\Gamma}, \bar{\kappa}_n)$ is transversal *and strictly admissible*, while still being $W^{2,2}$-close to (Γ, κ_n) with $(\bar{\Gamma}, \bar{\kappa}_n) \sim (\Gamma, \kappa_n)$. (In [5] we provide an example showing that before modification Γ need not be strictly admissible on any interval.) It turns out [5] that transversality plus strict admissibility imply that $(\bar{\Gamma}, \bar{\kappa}_n)$ can be isometrically extended to a larger domain, so now there is room to perturb the ruling without spoiling admissibility. To avoid heavy notation let us summarize Step 1 by assuming that (Γ, κ_n) is transversal and strictly admissible. *Step 2.* By carefully mollifying the curvatures κ and κ_n and defining the new curves by integration, one obtains a smooth approximation to (Γ, κ_n). It is well defined thanks to Step 1, but the boundary conditions will be spoiled. *Step 3.* One shows that if Γ is strictly admissible and transversal then arbitrary L^2-small perturbed versions $\hat{\kappa}^{\varepsilon}$, $\hat{\kappa}_n^{\varepsilon}$ (in the present case the mollified versions) of κ, κ_n can be 'corrected' by adding $\varphi, \psi \in C_0^{\infty}$ such that the curvatures $\hat{\kappa}^{\varepsilon} + \varphi$, $\hat{\kappa}_n^{\varepsilon} + \psi$ define a mapping $(\bar{\Gamma}, \bar{\kappa}_n)$ that is $W^{2,2}$-close to the original one, is again strictly admissible and moreover satisfies $(\bar{\Gamma}, \bar{\kappa}_n) \sim (\Gamma, \kappa_n)$. The proof of this statement is based on an implicit function argument that interprets (2), (3) as isoperimetric constraints on $\bar{\kappa}$ and $\bar{\kappa}_n$; in fact it also involves a reparametrization which we have omitted here for simplicity. See [5] for the details.

To prove Theorem 1 one shows (by means of a nontrivial argument [5]) that up to a small boundary layer, on each of the finitely many (see Section 2) connected components of $S \setminus C_{\nabla u}$ the developable mapping ∇u can be fully described by a finite number of simple ∇u-integral curves $\Gamma^{(i)} \in W^{2,\infty}([0, T^{(i)}]; S)$. Applying Lemma 2 to the restrictions $u|_{[\Gamma^{(i)}(0,T^{(i)})]} = (\Gamma^{(i)}, \kappa_n^{(i)})$ and subsequently gluing together the resulting smooth pieces we obtain a mapping with the properties stated in Theorem 1.

Acknowledgements

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