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Probability Theory/Numerical Analysis

Estimating multidimensional density functions for random variables in Wiener space

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Abstract

The Malliavin–Thalmaier (Springer Finance, Springer-Verlag, Berlin, 2006) formula was introduced for the simulation of high dimensional probability density functions. However, when this integration by parts formula is applied directly in computer simulations, we show that it is unstable. We propose an approximation to the Malliavin–Thalmaier formula. In this Note, we find the order of the bias and the variance of the approximation error, and we apply the Malliavin–Thalmaier formula for the calculation of Greeks in finance. ***To cite this article: A. Kohatsu-Higa, K. Yasuda, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Estimations des distributions de probabilité multidimensionnel pour des variables aléatoires dans l'espace de Wiener. La formule de Malliavin–Thalmaier a été présentée (Springer Finance, Springer-Verlag, Berlin, 2006) pour la simulation des fonctions de densité multidimensionnelles. Mais quand cette formule d'intégration par parties est appliquée directement pour la simulation sur ordinateur, nous prouvons qu'elle est instable. Nous proposons une approximation à la formule de Malliavin–Thalmaier. Dans cette Note, nous trouvons l'ordre du biais et la variance de l'erreur d'approximation. Nous appliquons la formula de Malliavin–Thalmaier pour le calcul des Grecques en Finance. ***Pour citer cet article : A. Kohatsu-Higa, K. Yasuda, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

The goal of the present article is to estimate, through simulations, probability density functions of multidimensional random variables using Malliavin Calculus and discuss some of its applications.

Usually, a result applied to estimate a multidimensional density is the classical integration by parts formula of Malliavin Calculus that is stated, for example, in Proposition 2.1.5 of Nualart [5]. However, this expression is not very efficient for computer simulation, that is, it has an iterated Skorohod integral. Recently, Malliavin and Thalmaier [4], Theorem 4.23, gave a new integration by parts formula that seems to alleviate the computational burden for simulation of densities in high dimension. We call this formula the Malliavin–Thalmaier formula. In this formula, one needs to

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simulate only one Skorohod integral instead of the previous multiple Skorohod integrals. But there is still a problem, that is, the variance of the estimator is infinite. Therefore we propose a slightly modified estimator that depends on a modification parameter h, which will converge to the Malliavin–Thalmaier formula as $h \to 0$. This will generate a small bias and a variance which is not infinite.

First we find the order of the bias and the variance of the approximation error. After obtaining these error estimations and the corresponding optimal parameter h, we apply the Malliavin–Thalmaier formula to finance, especially to the calculation of Greeks.

Details and proofs of the results given here can be found in Kohatsu-Higa et al. [3].

2. Malliavin–Thalmaier representation of multidimensional density functions

Assume that $d \geqslant 2$ is fixed through this paper. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_m) \in \{1, \dots, d\}^m$, $m \in \mathbb{N}$, we denote by $|\alpha| = m$ the length of the multi-index.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Suppose that $F = (F_1, \dots, F_d) \in (\mathbb{D}^\infty)^d$ is a d-dimensional random vector on $(\Omega, \mathcal{F}, P)$. Notations and terminologies can be found in Nualart [5].

Definition 2.1. Given the $\mathbb{R}^d$-valued random vector F and the $\mathbb{R}$-valued random variable G, a multi-index α and a power $p \geqslant 1$ we say that there is an integration by parts formula in Malliavin sense if there exists a random variable $H_\alpha(F; G) \in L^p(\Omega)$ such that

$$IP_{\alpha,p}(F,G): \quad E\left[\frac{\partial^{|\alpha|}}{\partial \mathbf{x}^\alpha} f(F) G\right] = E\big[f(F) H_\alpha(F;G)\big] \quad \text{for all } f \in C_0^{|\alpha|}(\mathbb{R}^d).$$

We represent the delta function by $\delta_0(\mathbf{x}) = \Delta Q_d(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^d$, where Δ means Laplacian. If $f \in C_0^2(\mathbb{R}^d)$, then the solution of the Poisson equation $\Delta u = f$ is given by the convolution $Q_d * f$ where the fundamental solution (also called Poisson kernel) Q_d has the following explicit form; $Q_2(\mathbf{x}) := a_2^{-1} \ln|\mathbf{x}|$ and $Q_d(\mathbf{x}) := -a_d^{-1} \frac{1}{|\mathbf{x}|^{d-2}}$ for $d \geqslant 3$. Here a_d is the area of the unit sphere in $\mathbb{R}^d$. The derivative of the Poisson kernel is $\frac{\partial Q_d}{\partial x_i}(\mathbf{x}) = A_d \frac{x_i}{|\mathbf{x}|^d}$, where $i = 1, \dots, d$, $A_2 := a_2^{-1}$ and for $d \geqslant 3$, $A_d := a_d^{-1}(d-2)$.

Related to the Malliavin–Thalmaier formula, Bally and Caramellino [1], have obtained the following result:

Proposition 2.2. *(Bally and Caramellino [1].) Suppose that for some* $p > 1$, $\sup_{|\mathbf{a}| \leqslant R} E[|\frac{\partial}{\partial x_i} Q_d(F - \mathbf{a})|^{\frac{p}{p-1}} + |Q_d(F - \mathbf{a})|^{\frac{p}{p-1}}] < \infty$ *for all* $R > 0$, $\mathbf{a} \in \mathbb{R}^d$. *If* $IP_{i,p}(F;G)$, $i = 1, \dots, d$, *holds then the law of* F *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}^d$ *and the density* $p_{F,G}$ *is represented as, for* $\hat{\mathbf{x}} \in \mathbb{R}^d$,

$$p_{F,G}(\hat{\mathbf{x}}) = E\left[\sum_{i=1}^d \frac{\partial}{\partial x_i} Q_d(F - \hat{\mathbf{x}}) H_{(i)}(F;G)\right].$$

Corollary 2.3. *If* F *is a nondegenerate random vector and* $G \in \mathbb{D}^\infty$, *then* (2.2) *holds for the probability density function of the random vector* F *at* $\hat{\mathbf{x}} \in \mathbb{R}^d$.

Remark 1. Note that $p_{F,G}(\hat{\mathbf{x}}) = E[G | F = \hat{\mathbf{x}}] p_{F,1}(\hat{\mathbf{x}})$. Therefore the expression in (2.2) corresponds to a density only in the case that $G = 1$. To avoid introducing further terminology, we will keep referring to $p_{F,G}(\hat{\mathbf{x}})$ as the 'density'.

3. Error estimations

In this section, we give the rate of convergence of the modified estimator of the density at $\hat{\mathbf{x}} \in \mathbb{R}^d$.

Definitions and Notations.

1. For $0 < h < 1$ and $\mathbf{x} \in \mathbb{R}^d$, define $|\cdot|_h$ by $|\mathbf{x}|_h := \sqrt{\sum_{i=1}^d x_i^2 + h}$.

2. For $i = 1, \ldots, d$, define the following approximation to $\frac{\partial}{\partial x_i} Q_d$, for $\mathbf{x} \in \mathbb{R}^d$, $\frac{\partial}{\partial x_i} Q_d^h(\mathbf{x}) := A_d \frac{x_i}{|\mathbf{x}|_h^d}$.
3. Then we define the approximation to the density function of F as; for $\mathbf{x} \in \mathbb{R}^d$,

$$p_{F,G}^h(\mathbf{x}) := E\left[\sum_{i=1}^d \frac{\partial}{\partial x_i} Q_d^h(F - \mathbf{x}) H_{(i)}(F; G)\right]. \tag{1}$$

The next result gives the order of the error of the approximation to the density:

Theorem 3.1. *Let F be a nondegenerate random vector and $G \in \mathbb{D}^\infty$, then for $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_d) \in \mathbb{R}^d$,*

$$p_{F,G}(\hat{\mathbf{x}}) - p_{F,G}^h(\hat{\mathbf{x}}) C_1^{\hat{\mathbf{x}}} h \ln \frac{1}{h} + C_2^{\hat{\mathbf{x}}} h + \mathrm{o}(h),$$

where $C_1^{\hat{\mathbf{x}}}$ and $C_2^{\hat{\mathbf{x}}}$ are constants which depend on $\hat{\mathbf{x}}$, but are independent of h. The constants can be written explicitly.

Formal Proof. We can write

$$p_{F,G}(\hat{\mathbf{x}}) - p_{F,G}^h(\hat{\mathbf{x}}) = A_d \sum_{i=1}^d \int_{\mathbb{R}^d} \left(\frac{y_i - \hat{x}_i}{|\mathbf{y} - \hat{\mathbf{x}}|^d} - \frac{y_i - \hat{x}_i}{|\mathbf{y} - \hat{\mathbf{x}}|_h^d} \right) E\big[H_{(i)}(F; G) | F = \mathbf{y}\big] p_{F,1}(\mathbf{y})\, \mathrm{d}y_1 \cdots \mathrm{d}y_d.$$

We separate the domain of integration to $|\mathbf{y} - \hat{\mathbf{x}}| < 1$ and $|\mathbf{y} - \hat{\mathbf{x}}| \geqslant 1$ and use spherical coordinates. For $|\mathbf{y} - \hat{\mathbf{x}}| < 1$ term, we apply Taylor expansion to $E[H_{(i)}(F; G) | F = \mathbf{y}] p_{F,1}(\mathbf{y})$ around $\hat{\mathbf{x}}$, and we have the order $h \ln \frac{1}{h}$ from this term. For $|\mathbf{y} - \hat{\mathbf{x}}| \geqslant 1$ term, we can obtain the order h easily. □

Next we compute the rate at which the variance of the estimator using Q_d^h diverges:

Theorem 3.2. *Let F be a nondegenerate random vector and $G \in \mathbb{D}^\infty$.*

(i) *For $d = 2$ and $\hat{\mathbf{x}} \in \mathbb{R}^d$,*

$$E\left[\left(\sum_{i=1}^2 \frac{\partial}{\partial x_i} Q_2^h(F - \hat{\mathbf{x}}) H_{(i)}(F; G) - p_{F,G}(\hat{\mathbf{x}})\right)^2\right] = C_3^{\hat{\mathbf{x}}} \ln \frac{1}{h} + \mathrm{O}(1),$$

where $C_3^{\hat{\mathbf{x}}}$ is a constant which depends on $\hat{\mathbf{x}}$, but is independent of h. The constants can be written explicitly.

(ii) *For $d \geqslant 3$ and $\hat{\mathbf{x}} \in \mathbb{R}^d$,*

$$E\left[\left(\sum_{i=1}^d \frac{\partial}{\partial x_i} Q_d^h(F - \hat{\mathbf{x}}) H_{(i)}(F; G) - p_{F,G}(\hat{\mathbf{x}})\right)^2\right] = C_4^{\hat{\mathbf{x}}} \frac{1}{h^{d/2-1}} + \mathrm{o}\left(\frac{1}{h^{d/2-1}}\right),$$

where $C_4^{\hat{\mathbf{x}}}$ is a constant which depends on $\hat{\mathbf{x}}$, but is independent of h. The constants can be written explicitly.

Remark 2. In particular, for $h = 0$ one obtains that the variance of the Malliavin–Thalmaier estimator is infinite.

4. The central limit theorem

Obviously when performing simulations, one is also interested in obtaining confidence intervals and therefore the central limit theorem is useful in such a situation. In what follows $\Rightarrow$ denotes weak convergence and the index $j = 1, \ldots, N$ denote N independent copies of the respective random variables.

Theorem 4.1. *Let Z be a random variable with standard normal distribution. And $(F^{(j)}, G^{(j)}) \in (\mathbb{D}^\infty)^d \times \mathbb{D}^\infty$, $j = 1, 2, \ldots$, are independent identically distributed $(d+1)$-dimensional random vectors and $F^{(j)}$, $j = 1, 2, \ldots$, are nondegenerate random vectors.*

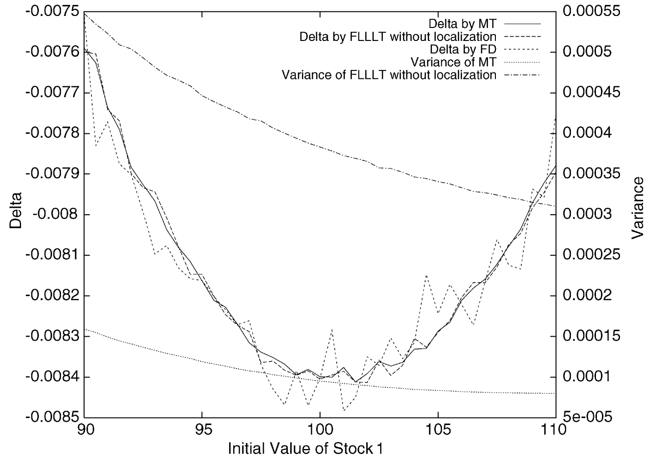

Fig. 1. Initial value of Stock 1 – Delta.

Initial value of $S_t^{(2)}$	100	Interest rate r	0
Drifts of $S_t^{(1)}$, $S_t^{(2)}$	0	Maturity T	1
Volatility of $S_t^{(1)}$	0.25	Strike prices K_1, K_2	100
Volatility of $S_t^{(2)}$	0.2	Number of MC	10^6
Correlation of B.M.	0.2	Number of time steps	8

Fig. 2. Parameters.

(i) *When $d=2$, set $n=\frac{C}{h\ln(1/h)}$ and $N=\frac{C^2}{h^2\ln(1/h)}$ for some positive constant C fixed throughout. Then*

$$n\left(\frac{1}{N}\sum_{j=1}^{N}\sum_{i=1}^{2}\frac{\partial}{\partial x_i}Q_2^h(F^{(j)}-\hat{\mathbf{x}})H_{(i)}(F;G)^{(j)}-p_{F,G}(\hat{\mathbf{x}})\right)\Longrightarrow\sqrt{C_3^{\hat{\mathbf{x}}}}Z-C_1^{\hat{\mathbf{x}}}C\quad as\ h\to 0,$$

where $H_{(i)}(F;G)^{(j)}$, $i=1,\dots,d$, $j=1,\dots,N$, denotes the weight obtained in the jth independent simulation (the same that generates $F^{(j)}$ and $G^{(j)}$).

(ii) *When $d\geqslant 3$, set $n=\frac{C}{h\ln(1/h)}$ and $N=\frac{C^2}{h^{d/2+1}(\ln(1/h))^2}$ for some positive constant C fixed throughout. Then*

$$n\left(\frac{1}{N}\sum_{j=1}^{N}\sum_{i=1}^{d}\frac{\partial}{\partial x_i}Q_d^h\big(F^{(j)}-\hat{\mathbf{x}}\big)H_{(i)}(F;G)^{(j)}-p_{F,G}(\hat{\mathbf{x}})\right)\Longrightarrow\sqrt{C_4^{\hat{\mathbf{x}}}}Z-C_1^{\hat{\mathbf{x}}}C\quad as\ h\to 0.$$

Remark 3. (i) In the assertion of Theorem 4.1, we can freely choose the constant C. Therefore, if C is small (wrt $C_1^{\hat{\mathbf{x}}}$), then the bias becomes small.

(ii) This theorem also gives an idea on how to choose h once n or N is fixed.

(iii) Numerical examples of the application of the above results appear in Kohatsu-Higa et al. [3].

5. Application of the Malliavin–Thalmaier formula to finance

We simulate the following Delta for a digital put option under two-dimensional Black–Scholes model;

$$\frac{\partial}{\partial S_0^{(1)}}E^Q\big[e^{-rT}\mathbf{1}\big(S_T^{(1)}\leqslant K_1\big)\mathbf{1}\big(S_T^{(2)}\leqslant K_2\big)\big],$$

where E^Q denotes an expectation under the risk-neutral measure Q, r is the constant interest rate of riskless asset, $S_T^{(1)}$, $S_T^{(2)}$ are stock values at the maturity T, $S_0^{(1)}$ is the initial value of $S_T^{(1)}$ and K_1, K_2 are strike prices. Parameters are given in Fig. 2. "Delta by MT" in Fig. 1 is computed through the Malliavin–Thalmaier formula (for details see Kohatsu-Higa et al. [3]). "Delta by FLLLT without localization" in Fig. 1 is computed by using the method of Fournié et al. [2]. And "Delta by FD" in Fig. 1 is computed by the two-sided finite difference method with bumping size 1.

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